REVISITING LANCHESTER'S SQUARE LAW FOR MILITARY STALEMATE

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INTRODUCTION

In a previous article [1], Maruszewski discussed the ideas behind Lanchester's Square Law for a military stalemate based on a differential equations model. Subsequently, Gordon, Gordon, *et al.* [2] developed the discrete analog of this model using difference equations. In both instances, however, the development used was dependent on algebraic arguments, but in the present article, we revisit this situation and utilize a geometric argument that is perhaps more intuitive.

THE DISCRETE MODEL

Suppose you are the head of the Joint Chiefs of Staff during a period of military action against the vile tyrant who is running the country of Gondwanaland. Our military intelligence experts report that the Gondwani Air Force has 100 jet fighters. The general commanding the operation believes that our fighter planes are four times better than those of the enemy because of our advanced technology and highly trained flight and support crews. How many planes should you authorize to be sent over into combat to be assured of air victory over Gondwanaland? Does 25 planes seem reasonable if we are supposedly four times better?

Let x_n represent the number of our jet fighters on the n^{th} day of the air war, and let y_n represent the number of enemy fighters. As mentioned above, our intelligence sources report that $y_0 = 100$. We seek to determine the value needed for x_0 that will guarantee at least a stalemate. Then, if we have more than x_0 planes at the start of the conflict, we should expect eventual victory. Of course, if we have fewer than this critical number, x_0, of planes, then we should expect eventual defeat.

It is reasonable to assume that the number of our planes lost on any given day is proportional to the number of enemy planes in the skies. Thus

$$\Delta x_n = -\beta y_n$$

where β is a constant of proportionality. Similarly, the number of enemy planes lost is proportional to the number of our planes in the air, so that

$$\Delta y_n = -\alpha x_n$$

where α is another constant of proportionality. This is a system of two linear difference equations

$$\Delta x_n = -\beta y_n \quad (1)$$

$$\Delta y_n = -\alpha x_n \quad (2)$$

or equivalently,

$$x_{n+1} = x_n - \beta y_n \quad (3)$$

$$y_{n+1} = y_n - \alpha x_n \quad (4)$$

where the initial conditions are

$$y_0 = 100, \quad x_0 = ?.$$

The belief that our planes are four times more effective than their planes means that $\frac{\alpha}{\beta} = 4$.

There are several ways to obtain closed form solutions to this system of difference equations. One approach is to "take the difference" of one of the two equations (1) and (2) and use the other equation to form a linear, second order difference equation with constant coefficients and to apply standard solution methods to solve such an equation. As shown in [2], the resulting solution is

$$x_n = C_1\left(1+\sqrt{\alpha\beta}\right)^n + C_2\left(1-\sqrt{\alpha\beta}\right)^n$$

$$y_n = -C_1\sqrt{\frac{\alpha}{\beta}}\left(1+\sqrt{\alpha\beta}\right)^n + C_2\sqrt{\frac{\alpha}{\beta}}\left(1-\sqrt{\alpha\beta}\right)^n$$

where

$$C_1 = \frac{1}{2}\left(x_0 - \sqrt{\frac{\beta}{\alpha}}y_0\right)$$

$$C_2 = \frac{1}{2}\left(x_0 + \sqrt{\frac{\beta}{\alpha}}y_0\right).$$

Alternatively, because the terms on the right side of the difference equations (3) and (4) are linear functions of x_n and y_n, it is possible to find the closed form solution to the system using standard matrix methods where the eigenvalues of the corresponding matrix turn out to be $1+\sqrt{\alpha\beta}$ and $1-\sqrt{\alpha\beta}$. Unfortunately, these explicit expressions do not tell us much about the behavior of the solutions or the progress of the air war. Nevertheless, after a rather complex algebraic argument, one can determine that in order for a stalemate to occur, it is necessary that

$$\left(\frac{x_0}{y_0}\right)^2 = \frac{\beta}{\alpha}. \tag{5}$$

This relationship is known as Lanchester's Square Law and any combination of x_0 and y_0 that fulfils this condition will lead to a stalemate.

In comparison, the corresponding continuous model is based on the system of differential equations

$$x' = -\beta y$$
$$y' = -\alpha x$$

whose solution has essentially the same form as the solution of the difference equation model. Furthermore, the same relationship (5) must hold in this continuous model in order to have an eventual stalemate.

We note, though, that there are some serious flaws in using differential equations to model a situation such as this where the process is inherently discrete. (From a pedagogical perspective, difference equation models can typically be analyzed using mathematics no more sophisticated than the precalculus level and often just introductory algebra, as in the current case, so that some exceptionally interesting and motivating applications can be made accessible to a much wider student audience.)

THE PHASE PLANE PORTRAIT

Instead of thinking of the number of jets x_n and y_n as two separate functions of n, we now consider how each is a function of the other. The resulting plot of the points (x_n, y_n), for $n = 0, 1, 2, \ldots$ is known as the phase plane portrait for the two solutions. The equilibrium solution for the system (1) and (2) corresponds to $\Delta x_n = \Delta y_n = 0$, and this occurs only when $x_n = y_n = 0$. Also, since the number of planes on both sides can never be negative, we consider only the first quadrant. For any point (x_n, y_n) in the first quadrant, the difference equations (1) and (2) tell us that $x_n < 0$ and $y_n < 0$, so all trajectories must move downward and to the left, as shown in Figure 1. This makes sense, since one can only lose planes in an air war, at least if we assume that neither side replenishes its losses. (Although this is one of

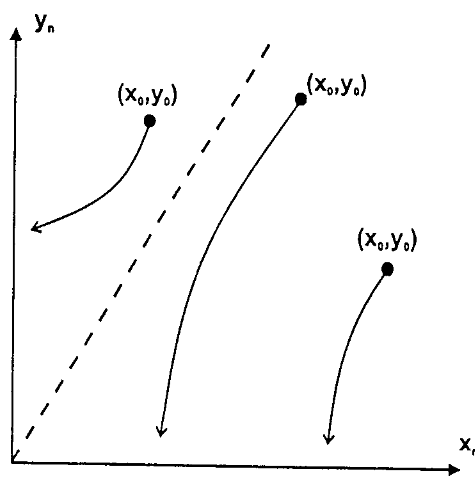

Figure 1

Lanchester's basic assumptions, it does seem a bit unrealistic.)

The actual patterns of the trajectories depend on the location of the initial point (x_0, y_0). For points relatively close to the x_n-axis, the y's decrease faster than the x's, so the path is concave down as it converges toward the x_n-axis, as shown in Figure 1. (The concavity will be demonstrated shortly) For points relatively close to the y_n-axis, the reverse is true and the path is concave up as the x's reduce faster than the y's.

For a stalemate, both air fleets should decrease at rates that just balance each other, so that the trajectory moves from the initial point (x_0, y_0) directly toward the origin. Let's see what combination of x_0 and y_0 values leads to this situation. Using the difference equations (3) and (4), we find that

$$x_1 = x_0 - \beta y_0$$
$$y_1 = y_0 - \alpha x_0.$$

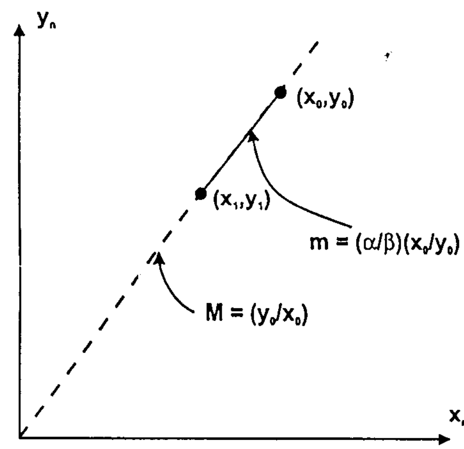

Figure 2

The slope of the line through the two points (x_0, y_0) and (x_1, y_1), as shown in Figure 2, is

$$m = \frac{\Delta y}{\Delta x} = \frac{-\alpha x_0}{-\beta y_0} = \left(\frac{\alpha}{\beta}\right)\left(\frac{x_0}{y_0}\right).$$

However, if these two points are to lie on a line through the origin, then the slope of that line must be equal to the slope of the line through the origin and (x_0, y_0), which is

$$M = \frac{y_0}{x_0}.$$

When we equate these two expressions for the slope, we obtain

$$\left(\frac{x_0}{y_0}\right)^2 = \frac{\beta}{\alpha},$$

which is the same relationship as in Equation (5) for Lanchester's Square Law.

Based on this relationship, we can now easily answer the question posed at the start of this article about the number of our planes needed to guarantee a stalemate against the 100 planes in the Gondwani Air Force.

Assuming that $\frac{\alpha}{\beta} = 4$ and $y_0 = 100$, we find that

$$\frac{x_0^2}{100^2} = \frac{1}{4}\text{, so that } x_0 = 50 \text{ planes}$$

to ensure a stalemate. Therefore if we send a minimum of 51 planes, we would be guaranteed an air victory over the Gondwanis (mathematically speaking).

Suppose now that we take the square root of both sides in Equation (5) to get

$$\frac{x_0}{y_0} = \sqrt{\frac{\beta}{\alpha}}$$

or equivalently

$$x_0 = \sqrt{\frac{\beta}{\alpha}}y_0. \quad \textbf{(6)}$$

For a given value of y_0, if the initial number, x_0, of x's is greater than the quantity on the right in Equation (6), then the y's will reduce more quickly and the x's will be victorious; on the other hand, for a given value of y_0, if

$$x_0 < \sqrt{\frac{\beta}{\alpha}}y_0.$$

then the x's will decrease more rapidly, and the y's will win. From a geometric perspective, we rewrite Equation (6) in the form

$$X = \sqrt{\frac{\beta}{\alpha}}Y \text{ or } Y = \sqrt{\frac{\alpha}{\beta}}X \quad \textbf{(7)}$$

and think of it as the equation of a line through the origin with slope $\sqrt{\frac{\alpha}{\beta}}$.

Based on our arguments above, it appears that any initial combination of x_0 and y_0 that lie on this line will lead to a stalemate. Similarly, any initial combination of x_0 and y_0 that lies above the line leads to a victory for the y's and any combination that lies below the line leads to a victory for the x's. Let's verify that these statements are indeed correct. We previously demonstrated that if the initial point (x_0, y_0) lies on this line, then the subsequent point (x_1, y_1) also lies on it. It is fairly easy to extend this line of reasoning - essentially an inductive argument - to show that if (x_n, y_n), for any n, lies on this line, so that

$$y_n = \sqrt{\frac{\alpha}{\beta}}x_n,$$

then the succeeding point (x_{n+1}, y_{n+1}) also lies on this line. Using the difference equations (1) and (2), we have $\Delta x_n = -\beta\, y_n$ and $\Delta y_n = -\alpha\, x_n$, so that the slope of the line through the two successive points is

$$m = \frac{\Delta y}{\Delta x} = \frac{-\alpha x_n}{-\beta y_n} = \frac{\alpha}{\beta}\left(\frac{x_n}{y_n}\right) = \frac{\alpha}{\beta}\left(\frac{x_n}{\sqrt{\frac{\alpha}{\beta}}x_n}\right) = \sqrt{\frac{\alpha}{\beta}}$$

and this is the same as the slope of the line in Equation (7) through the origin. Suppose now that the initial point (x_0, y_0) is below the line, so that

$$\frac{y_0}{x_0} < \sqrt{\frac{\alpha}{\beta}}$$

and hence

$$y_0 < \sqrt{\frac{\alpha}{\beta}}x_0 \quad \text{and} \quad x_0 > \sqrt{\frac{\beta}{\alpha}}y_0.$$

From the difference equation (3), we have

$$x_1 = x_0 - \beta y_0 > x_0 - \beta\sqrt{\frac{\alpha}{\beta}}x_0 = x_0\left(1-\sqrt{\alpha\beta}\right).$$

From the difference equation (4), we have

$$y_1 = y_0 - \alpha x_0 < y_0 - \alpha\sqrt{\frac{\beta}{\alpha}}y_0 = y_0\left(1-\sqrt{\alpha\beta}\right).$$

Therefore,

$$\frac{y_1}{x_1} < \frac{\left(1-\sqrt{\alpha\beta}\right)y_0}{x_1} < \frac{\left(1-\sqrt{\alpha\beta}\right)y_0}{\left(1-\sqrt{\alpha\beta}\right)x_0} = \frac{y_0}{x_0}$$

and so the point (x_1, y_1) is also below the line. Furthermore, the slope of the line segment connecting (x_1, y_1) to (x_2, y_2) is

$$m_2 = \frac{\alpha}{\beta}\left(\frac{x_1}{y_1}\right) > \frac{\alpha}{\beta}\left(\frac{x_0}{y_0}\right) = m_0,$$

where m_0 is the slope of the line segment connecting (x_0, y_0) to (x_1, y_1), so that the succeeding slope is steeper. Clearly, this argument can be extended inductively to show that each successive line segment, from right to left, has greater slope, so that the trajectory is concave down. In a similar manner, we can show that if any point lies above the line, the succeeding point is also above the line and the slope of the line segment connecting each successive pair of points on the trajectory is less steep than the slope of the preceding segment.

HOW FAST ARE PLANES LOST?

Having obtained results on the mathematical behavior of the two air fleets, we now turn to the question of how quickly do the fleets actually diminish? Equivalently, what are the human and equipment costs involved in an air victory? We again consider the example with $\frac{\alpha}{\beta} = 4$ and $y_0 = 100$. Suppose that we believe that our forces are capable of eliminating 8% of the Gondwani forces each day of hostilities, so that $\alpha = 0.08$, and therefore $\beta = 0.02$. Further, let's start with $x_0 = 50$, which will produce a stalemate. We obtain the set of successive values in the table to the right for the number of planes on both sides, where we have rounded the numbers obtained to the nearest integer. From these results, we see that the two air fleets slowly decimate each other and that the ratio of the number of remaining planes is roughly constant. Theoretically, if we did not round the successive results, the ratios would be identical and equal to ½, so that if one continues this process, the two air fleets would eventually simultaneously decay down to 0 each. This will take approximately 60 days.

x_n	y_n
50	100
48	96
46	92
44	88
42	84
40	81
38	78
36	75
35	72
34	69
33	66
32	63
31	60
30	58
29	56
28	54
27	52
26	50
25	48
...	...

What happens if we change the initial configuration to $x_0 = 51$ with $y_0 = 100$? We show the results in the table on the left. We observe that the results are approximately the same over the first 13 days, due to the rounding, and then the Gondwani forces decline slightly more rapidly and our forces decline slightly more slowly. After about 58 days, the model predicts that the Gondwani Air Force will be totally eliminated and that we would be left with 12 planes. From the viewpoint of our Chief of Staff, this is likely not very acceptable and so he would certainly assign a larger initial force to the conflict.

x_n	y_n
51	100
49	96
47	92
45	88
43	84
41	81
39	78
37	75
36	72
35	69
34	66
33	63
32	60
31	57
30	55
29	53
28	51
27	49
26	47
...	...

We show the results of different initial values for x_0 for comparison. First, if our side starts with $x_0 = 60$ planes, the results are illustrated in the table on the right. After 29 days, the Gondwani forces will be eliminated and we will be left with 33 planes.

x_n	y_n
60	100
58	95
56	90
54	86
52	82
50	78
48	74
47	70
46	66
45	62
44	58
43	54
42	51
41	48
40	45
39	42
38	39
37	36
36	33
...	...

If our side starts with $x_0 = 75$ planes, the results are illustrated on the top right. After 21 days, the Gondwani forces will be eliminated and we will be left with 55 of our original 75 planes.

x_n	y_n
75	100
73	94
71	88
69	82
67	76
65	71
63	66
62	61
61	56
60	51
59	46
58	41
57	36
56	31
55	26
55	22
55	18
55	14
55	10
...	...

If the two air forces have the same number of planes initially, then the results are illustrated on the left.

x_n	y_n
100	100
98	92
96	84
94	76
92	68
91	61
90	54
89	47
88	40
87	33
86	26
85	19
85	12
85	5
85	0

In this case, in just two weeks, the Gondwani air force is destroyed with the loss of 15 of our planes.

Thus, if we start with an overwhelming force of 150 planes, we get the results illustrated on the bottom right.

x_n	y_n
150	100
148	88
146	76
144	64
143	52
142	41
141	30
140	19
140	8
140	0

It is worth noting that in the conflicts in the skies over Iraq and Yugoslavia, neither enemy sent their planes up to fight. The forces arrayed against them were so great and their military leaders had the benefits of mathematical models such as this one, that it was clear that it would be more prudent to retain their planes. In fact, at the start of the Gulf War, Iraq sent 100 jet fighters over the border to Iran to keep them out of harm's way. Unfortunately for Iraq, the Iranians accepted the planes as a permanent gift.

Finally, according to an article in Newsday [3], official government estimates indicate that the United States' jet fighters such as the F-15 and the F-16 have a 5:1 advantage (the ratio $\frac{\alpha}{\beta}$) over such potential enemies as Russia, China, and North Korea, and an overwhelming 26:1 advantage over potential adversaries such as Iraq and Iran.

Explorations such as this example are easily implemented using a spreadsheet, and the author encourages interested readers and their students to investigate other scenarios in this context.

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