Functioning in the Real World:

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The reform of the calculus curriculum is well under way with changes in emphasis which reflect geometric and numerical ideas in addition to symbolic manipulations. There is also a greater emphasis on applications, the use of technology, and student projects. These new curricula have been adopted at hundreds of institutions around the country, and almost all the sites have reported success and satisfaction. More institutions are now thinking seriously about changing their calculus offerings. The results of these efforts may very well transform calculus into a *pump, not a filter.*

It is now time to consider how we "fill the tank"; that is, how we get more students into calculus. Each year approximately 600,000 college students take a precalculus/college algebra and trigonometry course; yet only about 15–20% of them ever go on to *start* calculus. (Admittedly, this includes many who take college algebra and trigonometry courses which are required terminal math courses; but we should expect a good course will turn on students to mathematics so that they would go on to calculus.) Most of those students who do go on to calculus from these traditional "preparatory" courses retain little of the material they were taught and do not complete calculus. This is a dreadful indictment of precalculus courses. They neither motivate the students to go on in mathematics nor adequately prepare them when they do continue.

What is needed is a precalculus experience that extends the common themes in most of the calculus reform projects, one that focuses more on mathematical concepts, one that provides students with an appreciation of the importance of mathematics in a scientifically oriented society, one that gives students the skills and knowledge they will need for subsequent mathematics courses, and one that makes appropriate use of technology.

The Math Modeling/PreCalculus Reform Project, with major funding from NSF, is addressing this problem by designing a new way to *fill the mathematics tank.* Our efforts involve developing, testing and implementing a dramatically different alternative to standard precalculus courses. Our goal is to emphasize the qualitative, geometric and computational aspects of mathematics within a framework of mathematical modeling at a level appropriate to precalculus students. We want to capitalize on the fact that most students are more interested in the applications of mathematics than in the mathematics itself, so that the applications should drive the mathematical development. Thus, all the mathematical knowledge and skills they will need for calculus are introduced, developed, and reinforced while applying mathematics to model and solve interesting and realistic problems. Our hope is that such an approach will excite the students and encourage them to go further with mathematics by showing them some of the payoff that a knowledge of mathematics provides.

The goal of our project is to develop a set of materials and a course based on them that will serve a multiplicity of audiences:

- A one semester course that will lay a different, but very effective, foundation for calculus;
- A one or two semester course that will stand as a contemporary capstone to the mathematics education of students who do not plan to continue on to calculus. We expect that the course will encourage many of these students to change their minds and to go on to calculus and other quantitatively related courses.
- A course that will provide the foundation for further courses in discrete mathematics and related offerings.

An outline of the materials developed is shown in Appendix I.

The models developed are based primarily on difference equations, data analysis, probability, and matrix algebra. The focus on difference equations includes applications of first and second order difference equations and systems of first order difference equations. The emphasis is on modeling a variety of situations and relating the solutions to the situations. For example, we develop an assortment of models on growth and decay processes (for populations, diseases, technology, etc.)

using both first order equations and systems of equations. Our treatment of second order linear difference equations includes a treatment of simple harmonic motion for a mass on a spring as well as the case of a system with damping. Other applications include projectile motion in the plane, effectiveness of sorting methods in computer science, and other models from physics, biology, economics, sociology and so forth.

As an example, consider the problem of modeling the way that the kidneys eliminate a medication from the bloodstream. Suppose a patient takes 50 mg of a certain drug each day and that the kidneys remove 80% of the drug in the patient's blood during each 24 hour period. Let D_0 represent the initial dosage of 50 mg. Then on the following days,

$$\begin{aligned} D_1 &= .20D_0 + 50 = 60 \text{ mg} \\ D_2 &= .20D_1 + 50 = 62 \text{ mg} \\ D_3 &= .20D_2 + 50 = 62.4 \text{ mg} \end{aligned}$$

and so forth. The successive values form an increasing sequence based on the first order difference equation

$$D_{n+1} = .20D_n + 50.$$

We can graph these values, connect the resulting points with a smooth curve, observe that the shape is concave down and that there is a limiting value (known as the maintenance level for the medication). This asymptote can be estimated visually, approximated numerically, or found in closed form. Later in the course, the difference equation can be solved in closed form, using techniques for solving any first order linear non-homogeneous difference equation to produce the specific solution satisfying the initial condition:

$$D_n = 62.5 - 12.5(.2)^n$$

for all n. At each stage, thought-provoking questions are raised: What happens if the patient takes an overdose (a dose above the maintenance level)? or Is the smooth curve used to connect the successive points reasonable?

The techniques used to solve problems involving difference equations are the direct analogs of the techniques used for differential equations. For example, consider what a student must do to solve a second order linear difference equation: The student must find the roots of an associated characteristic equation (using factoring, the quadratic formula, and possibly numerical methods) to construct the general solution of the homogeneous case. He or she must then construct a trial solution for the nonhomogeneous case which involves recognizing classes of functions, performing extensive algebraic manipulations with indices and exponents, and solving systems of simultaneous linear equations. Imposing initial conditions involves solving further systems of linear equations. Most importantly, though, the student must demonstrate an understanding of what information the solution provides about the behavior of the process being studied, particularly in a geometric or qualitative way.

To illustrate this procedure, consider the second-order linear nonhomogeneous difference equation

$$x_{n+2} + 5x_{n+1} + 6x_n = 21 \cdot 4^n.$$

The corresponding homogeneous difference equation is

$$x_{n+2} + 5x_{n+1} + 6x_n = 0,$$

whose associated characteristic equation is the quadratic equation

$$r^2 + 5r + 6 = 0,$$

having roots $r = -2$ and $r = -3$. Therefore, the general solution of the homogeneous equation is

$$x_n = C_1 \cdot (-2)^n + C_2 \cdot (-3)^n,$$

where C_1 and C_2 are any two arbitrary constants.

To find a particular solution of the nonhomogeneous equation, we note that the right-hand side is an exponential function, 4^n. Using a discrete form of the method of undetermined coefficients, we assume that the particular solution is also a multiple of 4^n and therefore try

$$x_n = A \cdot 4^n,$$

so that

$$\begin{aligned} x_{n+1} &= A \cdot 4^{n+1} = 4A \cdot 4^n \quad \text{and} \\ x_{n+2} &= A \cdot 4^{n+2} = 4^2 \cdot A \cdot 4^n = 16 \cdot A \cdot 4^n. \end{aligned}$$

Substituting these quantities into the original difference equation, we find

$$x_{n+2} + 5x_{n+1} + 6x_n = 42A \cdot 4^n = 21 \cdot 4^n,$$

so that $A = \frac{1}{2}$. The complete solution to the original difference equation is

$$x_n = C_1 \cdot (-2)^n + C_2 \cdot (-3)^n + \frac{1}{2} \cdot 4^n.$$

Since this solution involves two arbitrary constants, C_1 and C_2, we require two initial conditions, x_0 and x_1 to determine a specific solution uniquely. Finally, from the fact that 4^n eventually dominates any multiple of $(-2)^n$ or $(-3)^n$, we conclude that the overall behavior of the solution will be one of exponential growth after some possible initial oscillation. The behavior pattern can easily be verified graphically using a graphing calculator or a computer graphics program which displays the solution. It is also possible to examine the first few terms of the solution directly from iterating the original difference equation based on the given initial values.

Furthermore, second order difference equations such as this often lead to pairs of complex roots for the characteristic equation. In turns, this requires that the student uses complex numbers and DeMoivre's theorem in the kind of setting that we eventually want them applied; this is quite different from standard precalculus courses where DeMoivre's theorem is briefly introduced, but never used again until the middle of a course in differential equations several years later.

As another instance, many of the standard formulas for sums of special terms, such as $\sum n$, $\sum n^2$ and $\sum r^n$ can all be developed as simple difference equations whose solutions can be found directly. Thus, rather than handing the students a set of formulas (with no motivation as to where they come from) and then proving them with a poorly understood induction argument, we provide a framework for solving much more varied problems. Further, because of the emphasis on applications, these formulas are then used repeatedly in the sequel to reinforce them.

Data analysis techniques are also introduced early and used throughout so that students learn how to interpret data values that arise in many different contexts. For instance, ideas on regression analysis are developed, including nonlinear regression as a way of reinforcing notions on the behavior of different classes of functions. The computational drudgery is relegated to a computer or graphing calculator with statistical functions. Further, the focus is on collecting and analyzing real data sets and matching them to appropriate mathematical models and interpreting the results.

We expect the students to be actively involved in the process of transforming the data to linearize it. Initially, we require them to identify the likely behavior pattern(s) from a scatterplot, to perform the appropriate transformations by hand, to obtain the best linear fit to the transformed data using their technological tools, and then to undo the transformation by hand. This typically involves extensive manipulations with exponential and logarithmic functions and their properties. Finally, we interpret the results of each analysis and ask appropriate questions about predicting (interpolation and extrapolation) as well as questions about when a given level is expected to be reached. Eventually, we do relegate most of the transformations to the technology, but not before the students understand what is happening and have had the opportunity to reinforce their mechanical skills.

Another thread which is interwoven throughout the course(s) is the notion of probabilistic ideas in the context of performing random simulations. These include Monte Carlo simulations to estimate the value of π, to estimate the average value of a function on an interval, to estimate the frequency of complex roots of polynomials, to study essentially random processes such as radioactive decay and the spread of diseases, and to develop models to investigate various waiting-time situations.

In particular, we emphasize geometric probability as a vehicle for reinforcing geometric and trigonometric ideas in a new setting. Typically, if a random variable is uniformly distributed, the probability of an event can be expressed as a ratio of two lengths, two time intervals, two areas or two volumes. For example, suppose a painting whose area is 144 square inches is framed by wood 2 inches wide. Someone is painting the ceiling in that room, but doesn't bother to remove the picture. The conditional probability that a drop of paint hitting the picture or frame misses the picture itself depends on W, the width of the picture. Find this probability as a function of W and graph this function. Estimate the dimensions of the picture which has the greatest probability of being hit by a paint spatter.

As a somewhat more sophisticated application, we introduce the idea of finding the area of a region under a curve using Monte Carlo simulations. As one application, we consider the area under the parabola $y = x^2$ from 0 to b. Geometrically, we demonstrate that this region lies inside a triangle using the maximum value of the function, so that its area is less than $\frac{1}{2}b \cdot b^2$. We then perform Monte Carlo simulations to estimate the area corresponding to $b = 1, 2, 3, 4, 5$, say, and use the values so obtained as a set of data to be analyzed. On one set of runs, the data was best fit by the power function

$$A = .333197 \cdot b^{3.0006}$$

with a correlation coefficient of $r = 0.999992$.

In general, our focus is on the *use* of probability as a *tool,* not on a formal treatment nor on standard balls-in-urns problems. We want to demonstrate to students that stochastic ideas and methods provide a valuable tool for investigating nondeterministic phenomena. In turn, we expect that such an applied introduction to the subject will motivate them and will provide the background necessary to make a full probability course more accessible.

Trigonometry is approached from the point of view of modeling periodic phenomena such as the number of hours of daylight in a given location as a function of the day of the year, the tides, the heart and so forth. We put a lesser emphasis on manipulation of trig functions for their own sake. Further, we use the trig functions as a vehicle for exposing the students to the ideas of approximating one function by another via "soft" introductions to the notions of Taylor polynomial approximations and Fourier approximations in graphical and numerical settings.

Matrix algebra is introduced not just as a shortcut method for solving systems of linear equations. There is an incredibly rich array of applications of the subject; we show these applications to students in a finite mathematics course, but rarely give even a glimpse of them to students in the mainstream math/science/engineering track. Yet, these are the applications that make the subject come alive to most students. Thus, we introduce matrix algebra as a unifying tool for investigating a wide array of applications including systems of difference equations, Markov processes, ~~regression analysis~~, geometry ~~and networks~~.

Our course also involves much computer or graphing calculator work to explore mathematical models. It involves many classroom experiments to investigate the accuracy of the mathematics in predicting the results of actual processes or to help develop mathematical models based on experimental data. For instance, Newton's Law of Cooling, the damped spring, and Torricelli's Law on fluid leaking out of a cylinder are treated experimentally as well as theoretically.

The course also features a series of student investigations to provide a real-life dimension to the mathematics. For example, we have students conduct individual investigations of mathematical ideas and methods using computer software and/or graphing calculators. We also have students collect sets of data of interest to *them*, say on some growth process or on the acceleration times of a Porsche, and eventually determine the best fitting curve, possibly exponential, logistic or power. Similarly, they have been asked to model the number of hours of daylight in a city of their choice based on actual data collected from the newspapers or other resources.

We find that this direct involvement in the mathematics and the wide applicability of the subject is providing students with the motivation and impetus to continue on to study mathematically related fields. This certainly is reflected in the comments we have received from students in the course. See Appendix II for a set of all the verbatim written comments from the students in one of the pilot versions of the course.

The PreCalculus Reform Project is being funded under a multiyear grant from the National Science Foundation under the joint direction of the author and B. A. Fusaro of Salisbury State University. The initial project working group includes Florence Gordon, Walter Meyer, Jim Sandefur (on leave to work at NSF in 1993), Martha Siegel, and Alan Tucker. We have since added Ellen Shatto and Judy Broadwin to the team to provide expertise on the high school perspective, since we feel that our course would be valuable at that level as well. The project directors have offered pilot versions of the course during the Spring 1992, Fall 1992 and Spring 1993 semesters; broader class testing has begun in Spring, 1993 and will be extended in the coming years. We plan to provide workshops to familiarize instructors with the project materials and prepare them to teach the course.

Appendix III includes a copy of the final exam given by the author during one of the pilot offerings of the course. Despite the sophistication of the problems, the students performed extremely well, particularly in comparison to the author's experiences with more traditional precalculus courses.

Interested readers are encouraged to contact the author or Fusaro for additional information on the details of the project. In particular, we welcome hearing from potential class testers of the course.

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Appendix II: Student Survey Responses

"My overall reaction to the course was extremely positive. By emphasizing the value of mathematical pursuits through applications first (and theory being derived from the application), the course proved to be constantly interesting. Past math courses seemed tedious. This course never struck me as tedious (challenging? very, but not tedious) Strong points of the course:

1. The professor's enthusiasm, approachability and good cheer.
2. The small class size made asking questions easy.
3. Most of the examples provided in the text were appropriate.
4. The text was easy to follow (with 1 or 2 exceptions).
5. Above all, being able to see how I could use the subject I was learning in the 'real' world.

Weaker points:

1. The section on Hooke's Law and Newton's Law of springs was tough to follow.
2. Computer material is tough to follow unless the student is the operator. So, my only suggestion would be: every other Thursday have a computer lab. Then, instead of testing us on regression analysis, we could be tested during lab practicals. This grade could count as one question on each exam. Nice to think of math as an intellectual pursuit, a very useful tool, and a way of seeing live—as opposed to 'a course I have to take to complete a chemistry curriculum.' I plan to take calculus through analytic geometry since I'm pursuing chemistry/environmental engineering. My plans have remained the same, but my outlook has certainly improved (By the way, the course somehow managed to cure my phobia of college physics.)"

"I think this course was great. It was probably the most enjoyable math class I ever took. I liked mostly the adding up the numbers in a sequence with $\frac{1}{2}n(n+1)$. It's fun to go home and ask my parents to add up the numbers between one and 1 hundred and they look at me like I'm nuts. And I can do it in a few seconds. Also, the $(1-r^{n+1})/(1-r)$ was very interesting. Also, I liked the Newton's law of cooling stuff like figuring out how long a body's been dead and cool stuff like that. And I am not a kisser but cause my name isn't on this but I have never had any teacher as good as you. And I hope that the next calculus classes are as good as this one. I was always strong at math but this course has made it enjoyable and has given a much more positive attitude towards calculus."

"I liked this course a lot. However, it was a very hard course. We had a very smart teacher that knew how to get down to a student's level and help us understand. It was interesting how much math has to do with real life. Math can really be applied to real life. I see that math is used throughout real life. I wish that my next calculus course would be taught in this way. I am planning to take calculus I and II but now I'm afraid that those courses will be taught in the old way again when I just learned the new precalculus."

"My reactions to this course is a lot better now than it was in the beginning. When I took the previous course I didn't do well. I came into this course with the same attitude, but it has changed. I feel that Difference Equations was the bulk of this course. We did a lot of work for Difference Equations and I now find them fairly easy. Difference Equations are definitely most interesting because of the amounts of time put into it. I also think it has a lot to do with the teacher, I found my teacher very helpful which made the course interesting.

I have always liked math, but the course before this one changed my mind. I didn't like math anymore and found it very hard. But this course changed my mind again. I like math and I find it interesting."

"The sheets were very helpful & followed the structure of the course. Also, I appreciated the fact that the problems were relevant to real life situations. This course was, however, very demanding. I regret that I am taking so many courses this semester and could not allot more time to this one. The work with the computer I found helpful; however, it was confusing to load the programs. . . . I can no longer harp that math is irrelevant to the 'real world' because it has been proved to me that it is indeed relevant."

"I felt that this course was very interesting. Prof. Gordon did a great job teaching us the course. Relating math to real life situations is really good. I hate doing word problems, but the way it was represented gave me a chance to do them and maybe somewhat like them. I was always one to just do the problems given the numbers and letters, but this course has helped me be able to figure out an answer by not really having too many numbers and letters. I have always enjoyed math but this course made it just a little bit more fun."

"I wasn't used to the way this class was taught so I found it hard to study for. Answers to the homework problems would definitely be beneficial. I thought the applications to the difference equations were interesting, I liked to see how the mathematics applied to real life. Overall, I liked this class better than any other math class I've taken. This course made me see that math is not just a bunch of numbers and graphs. Math is used in everything even if it is not obvious."

"In a way, I liked this course very much (it was something different), at the same time I believer that I could do better with answers on the back of the book. I think that many things in this course is very interesting and I know that I wouldn't learn them in a regular MA 62 class. I hope more courses will be taught this way and that many students will enjoy it as much as I did."

"I liked the applications to realistic terms, not infinite problems but ones with limits. Finding solutions to difference equations is extremely hard and it went a little to fast. Computer projects are time consuming and frustrating.

In physics we found the data in experiments and gave a quick equation to it. But in here we worked with equations and found experiments that fit. I may be a little ahead now for my science classes."

MA 62 Final Exam

December 17, 1992

1. a) Find the complete solution to the difference equation

$$x_{n+1} - 3x_n = 12n - 8$$

b) If $x_0 = 2$, find the corresponding solution.

c) Calculate the first 5 terms of the solution using the difference equation.

d) Calculate the first 3 terms of the solution using the solution from part b).

e) Draw a very rough sketch of the solution to show its behavior.

2. Solve each of the following difference equations:

a) $x_{n+2} + 3x_{n+1} - 10x_n = 6 \cdot 3^n$

b) $6x_{n+2} - 5x_{n+1} + x_n = 14$

Draw a very rough sketch of each solution to show its behavior.

3. Given $x_{n+2} - 8x_{n+1} + 20x_n = 0$.

a) Find the general solution.

b) What are the modulus and angles for the complex roots?

c) Convert the general solution into the equivalent solution in trigonometric form.

4. A certain radioactive element decays 20% each year. If 25 grams are present initially,

a) what is the amount present after 10 years?

b) what is the half life?

5. The kidneys remove 80% of any impurity from the blood each day. A woman is prescribed a daily medication with a dose of 100 mg each day.

a) Solve the difference equation for the level of drug in her system daily.

b) What is the level of dosage in her system after 10 days?

c) What is the maintenance level of the dosage?

6. The United States produced 2.5 million barrels of crude oil in 1985 and production has been increasing at an annual rate of 1.7%. If this trend continues, what is the total U.S. oil production between 1985 and 2000?

7. A blacksmith takes a horseshoe out of a furnace at 1200° and dunks it into a barrel of water at 70°. After 10 minutes, the temperature of the horseshoe is 200°.

a) What is the temperature after 15 minutes?

b) How long does it take until the temperature reaches 80° so that the blacksmith can take it out of the water and put the shoe on a horse?

(According to Newton's Law of Cooling: $T_n = (T_0 - R)(1 + \alpha)^n + R$)

8. A young couple sets up a savings plan based on a 4% annual interest rate. They deposit $1000 initially and agree to increase the annual contribution by 10% each year.

a) Express the above situation as a difference equation.

b) Find the solution to this difference equation.

c) Sketch a very rough graph of the solution to show its behavior.

9. Given the points $A(1, -2)$ and $B(7, 16)$.

a) Find the equation of the line through A and B.

b) Find the midpoint of the line segment from A to B.

c) Find the point which is 3/5 of the way from A to B.

d) Find the distance from A to B.

10. The following graph shows the velocity of a plane as a function of time.

Indicate the following:

a) all intervals where the velocity is decreasing.

b) all intervals where the plane is accelerating.

c) all intervals where the curve is concave up.

d) all intervals where the curve is concave down.

e) all points where the velocity is greatest.

f) all points where the velocity is increasing most rapidly.

11. At a certain pier, the low water line is 8 feet above sea bottom and the high water line is 14 feet above bottom. If low tide occurs at midnight and high tide at 6 A.M., then the height of the water as a function of time t is $H = 11 + 3 \cdot \sin(2\pi(t-3)/12)$.

a) How high is the tide at 10 am?

b) When is the water 12 feet deep?

12. The perihelion and aphelion distances for Mercury are 35.4 and 36.0 million miles, respectively. Find the equation of Mercury's orbit about the sun.

13. Given $z = \|z\|(\cos\theta + i\sin\theta)$ and $z^2 = \|z\|^2(\cos 2\theta + i\sin 2\theta)$, show that $z^3 = \|z\|^3(\cos 3\theta + i\sin 3\theta)$. (Hint: $z^3 = z \cdot z^2$.)

14. A biologist is attempting to develop a mathematical model relating a person's height (in inches) to his or her weight (in pounds) from infancy to maturity. She suspects the relationship involves a power law. The data values for one person studied are:

Wt:	5	17	24	33	46	59	73	98	120	195
Ht:	19	26	31	40	46	52	57	66	70	75

a) Draw the scatterplot for this data and indicate the regression line by eye.

b) Transform this data to linearize it.

c) Draw the scatterplot for the transformed data and indicate its regression line. Suppose the computer produces the following linear regression equation for the transformed data:

$$Y = .43X + .95$$

d) De-transform this equation to produce the exponential function which best fits the data.

e) What is your prediction for the person's height when he weighed 140 pounds?

f) What is your prediction for the person's height if he weighed 220 pounds?

g) Which prediction would you have more confidence in? Why?

15. Transform $\sin^4 x$ into an equivalent expression that does not contain any powers.