provided differentiation of the 'Kronecker components', v^a, is taken to be ordinary partial differentiation. In the same manner, for a covariant vector one obtains

$$u_{a;b} = (u_c\delta^{\bar{c}}_a)_{;b} = u_{c,b}\delta^{\bar{c}}_a + u_c\delta^{\bar{c}}_{a;b} = u_{a,b} - u_c\Gamma^c_{ab},$$

again provided differentiation of the 'Kronecker components', u_c, is taken to be ordinary partial differentiation. In other words, if Levi–Civita differentiation is defined for Kronecker rows and columns, it may be extended to all tensors by means of the product rule.

Moreover, since $\delta^{\bar{c}}_{a;b}$ and $\delta^c_{\bar{a};b}$ are basically different, we suspect that $-\Gamma^c_{ab}$ is not an affine connection. To check this, start from the official change law,

$$\Gamma^\gamma_{\alpha\beta} = D^a_{\alpha\beta}D^\gamma_a + \Gamma^c_{ab}D^a_\alpha D^b_\beta D^\gamma_c$$

In the critical first term, there is a second derivative of Latin coordinates with respect to Greek, and a first derivative of Greek with respect to Latin. Call this, of type $K\,J^{-1}$. Now multiply by -1, to obtain

$$-\Gamma^\gamma_{\alpha\beta} = -D^a_{\alpha\beta}D^\gamma_a + (-\Gamma^c_{ab})D^a_\alpha D^b_\beta D^\gamma_c$$

For an unbiased comparison, a plus is needed in the first term. It can be obtained from

$$0 = (D^a_\alpha D^\gamma_a)_{,\beta} = D^a_{\alpha\beta}D^\gamma_a + D^\gamma_{ab}D^a_\alpha D^b_\beta$$

Thus, $-D^a_{\alpha\beta}D^\gamma_a = D^\gamma_{ab}D^a_\alpha D^b_\beta$, and the law now reads

$$-\Gamma^\gamma_{\alpha\beta} = D^\gamma_{ab}D^a_\alpha D^b_\beta + (-\Gamma^c_{ab})D^a_\alpha D^b_\beta D^\gamma_c$$

Therefore, $-\Gamma^c_{ab}$ is of type $K^{-1}J^2$, emphatically not an affine connection. And any M^c_{ab} of type $K^{-1}J^2$ would serve.

To sum up, the usurpers $\delta^{\bar{c}}_{a;b}$ and $\delta^c_{\bar{a};b}$ have taught us that a 'connection' is really a pair, $(\Gamma^c_{ab}, M^c_{ab})$, one of type KJ^{-1} for differentiating contravariant vectors, and the other of type $K^{-1}J^2$ for differentiating covariant vectors.

5. Conclusion

Well, should not the tocsin be sounded? With Jacobi and Levi–Civita fallen, Cantor himself may be in peril. Can we be expelled from Paradise, despite Hilbert? Will the Ghost of Kronecker drive us all mad?

As the chestnut forest tries to overcome the blight, even by subterranean modes, so everywhere we little men will be rearranging deltas until something gives.

Don't hoard a secret of this gravity. Tell your neighbour; hire a horse and lantern. The delta-coats are coming!

We interrupt normal programming to bring you the following urgent message. . . .

l'Hopital's Rule and Taylor Polynomials

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The cornerstone of calculus is the concept of limit. Unfortunately, it is the least understood notion for most students in introductory calculus. Thus, when they are first exposed to limits at the beginning of the course, most students never see beyond the typical algebraic manipulation and cancellation that allows them to come up with the right answer. This is exacerbated later in the course when they are presented with that magic wand known as l'Hopital's Rule[1] which allows them to determine limits of all sorts of indeterminate forms without giving any thought to what the limit means.

The Harvard Calculus Reform project is intended to redefine the content and the spirit of introductory calculus and so revitalize the subject. Our feeling is that this requires achieving a proper balance between three different ways of looking at calculus—graphical, numerical and symbolic. For too long, calculus courses and the textbooks on which they are based have emphasized symbolic manipulation to the point where the other two perspectives have become minimal. Thus, for most students, the focus of calculus has been on obtaining the identical closed form solution to the problems that appear in the back of the book. We believe that by emphasizing all three aspects (what we call the *Rule of Three*), students will develop a far deeper understanding of the concepts, the applicability and the grandeur of calculus. See [1] for a more detailed description of the project.

In the present article, we will see how this philosophy can be applied to enhance student understanding of the limit process. In particular, we will indicate how limits can be reinterpreted from the point of view of Taylor polynomial approximations to give a much deeper insight into *why* the limit of a function is the value obtained using the manipulative techniques.

First of all, every limit problem should be accompanied by a brief computational investigation. This can be done in two ways. Before applying the usual manipulative techniques, students should be encouraged to estimate the value of the limit by calculating the value of the function for several values of the variable near the limit point using a calculator. Alternatively, after the limit has been formally obtained, they should be encouraged to check it out numerically by calculator to see that the values actually converge to the limit. In either way, they develop a far deeper appreciation of what the limit is. It no longer is just the right number at the back of the book; rather, it is the limiting value for the function as x approaches a. In this way, they achieve an emotional conviction for the accuracy of the result as well as a purely intellectual one.

This type of investigation can be extended by using either a graphing calculator or a computer graphics program. Have the students graph the indicated function, say $\sin x/x$, in some neighbourhood of the limit point and then zoom in on the desired point. Let them see that there is a corresponding value for the limit, or that there is clearly no limit as the point is approached. This approach naturally extends the ideas suggested by Steman in [2].

The above methods are essential in conveying to the student the idea of *what* the limit of a function is. Unfortunately, they do not easily convey understanding of *why* a particular limit is the value obtained. To do this requires a deeper understanding of the behaviour of the function under investigation near the limit point. This can be accomplished very easily once the notion of a Taylor polynomial approximation is available. In fact, the entire subject of Taylor approximations can be introduced at a much earlier stage of the course with some extremely useful benefits; see [3].

[1] The name *l'Hopital's Rule* is actually a misnomer. The rule was discovered by Johann Bernoulli. l'Hopital bought it from Bernoulli, published it in his 1696 Calculus textbook and so got the credit for the rule.

In a related direction, Mathews [4] discusses the use of the computer algebra system Mathematica as a means for determining limits through the use of Taylor polynomials. However, his emphasis is again too much on the *what* and the *how* of the limits and not enough on the *why*.

Most of the standard limit problems in introductory calculus courses can be treated extremely effectively using Taylor polynomials to help students see why the limit is the limit. For example, consider the limit of $\sin x/x$ as x approaches 0. We approximate the sine function with the first few terms of its Taylor expansion and so obtain

$$\sin x/x \approx (x - x^3/3! + x^5/5!)/x = 1 - x^2/6 + x^4/120$$

If we now allow x to approach 0, it is clear that the limit will be 1. The advantage to this approach is that the behaviour of the transcendental function is replaced by the simpler behaviour of the polynomial function in the neighbourhood of the limit point. Therefore, in this neighbourhood where the approximation is effective, the function is obviously going to behave as if it were simply $1 - x^2/6 + x^4/120$. Of course, this could be treated using a Taylor approximating polynomial of any desired degree.

Some further examples

Consider the following:

1. $(\sin 5x)/(3x) \approx [(5x) - (5x)^3/3! + (5x)^5/5!]/(3x)$

$$= (1/3)[5 - 5^3x^2/3! + 5^5x^4/5!]$$

and this clearly approaches 5/3 as x approaches 0.

2. $(1 - \cos x)/x^2 \approx [1 - (1 - x^2/2! + x^4/4! - x^6/6!)]/x^2$

$$= \tfrac{1}{2} - x^2/4! + x^4/6!$$

and this clearly approaches $\frac{1}{2}$ as x approaches 0. There is no need to apply the highly artificial l'Hopital's Rule twice just to obtain the answer, but not to convey any understanding.

3. $[\exp(x) - 1]/x \approx [(1 + x + x^2/2 + x^3/6) - 1]/x$

$$= 1 + x/2 + x^2/6 \to 1 \text{ as } x \to 0.$$

4. Suppose that it has been established that

$$\lim_{x\to 0} x^x = 1$$

Then using the linear approximation to the sine function, we obtain

$$\lim_{x\to 0} (\sin x)^x = \lim_{x\to 0} (x)^x = 1$$

also.

We can also use some variations of these ideas to investigate several other standard limits. For instance, consider

$$\lim_{n\to\infty} (1 + x/n)^n$$

for any value of x. We expand the above expression using the binomial theorem to obtain

$$(1+x/n)^n \approx 1+n(x/n)+\tfrac{1}{2}n(n-1)x^2/n^2+(1/3!)n(n-1)(n-2)x^3/n^3+\dots$$
$$=1+x+(x^2/2)(n-1)/n+(x^3/3!)[(n-1)/n][(n-2)/n]+\dots$$

which approaches

$$1+x+x^2/2+x^3/3!+\dots$$

as n approaches ∞. That is,

$$\lim_{n\to\infty}(1+x/n)^n=\exp(x)$$

In a similar way, we can show that

$$\lim_{x\to n}(1+1/x)^x=\mathrm{e}$$

Deriving l'Hopital's Rule

The above ideas can be expanded into an interesting and yet relatively simple proof of l'Hopital's Rule in the restricted case where $f(a)=g(a)=0$. We seek to determine the limit:

$$\lim_{x\to a} f(x)/g(x)$$

if it exists.

Suppose that both $f(x)$ and $g(x)$ possess derivatives up to order n, for some $n>1$, at the point $x=a$. We can then approximate each one of them by its corresponding Taylor polynomial approximation of degree n at a:

$$f(x)\approx f(a)+f'(a)(x-a)+f''(a)(x-a)^2/2!$$
$$+\dots+f^{(n)}(a)(x-a)^n/n!$$
$$g(x)\approx g(a)+g'(a)(x-a)+g''(a)(x-a)^2/2!$$
$$+\dots+g^{(n)}(a)(x-a)^n/n!$$

and so, since $f(a)=g(a)=0$, their ratio is simply

$$\frac{f(x)}{g(x)}\approx\frac{f'(a)(x-a)+f''(a)(x-a)^2/2!+\dots+f^{(n)}(a)(x-a)^n/n!}{g'(a)(x-a)+g''(a)(x-a)^2/2!+\dots+g^{(n)}(a)(x-a)^n/n!}$$
$$=\frac{f'(a)+f''(a)(x-a)/2!+\dots+f^{(n)}(a)(x-a)^{n-1}/n!}{g'(a)+g''(a)(x-a)/2!+\dots+g^{(n)}(a)(x-a)^{n-1}/n!}$$

Therefore, in the limit as x approaches a, we obtain

$$\lim_{x\to a} f(x)/g(x)=f'(a)/g'(a)$$

provided that $g'(a)$ exists and $f'(a)$ and $g('(a)$ are not both 0.

Moreover, if $f'(a)=g'(a)=0$, then it is clear that the original limit can be evaluated in terms of the ratio of the second derivative terms, and so forth.

Alternatively, we could derive l'Hopital's Rule precisely by using Taylor's Theorem with the Lagrange form or the remainder. Thus

$$\frac{f(x)}{g(x)}=\frac{f'(a)(x-a)+f''(a)(x-a)^2/2!+\ldots+f^{(n)}(a)(x-a)^n/n!+R_n}{g'(a)(x-a)+g''(a)(x-a)^2/2!+\ldots+g^{(n)}(a)(x-a)^n/n!+S_n}$$

where

$$R_n=f^{(n+1)}(C_f)(x-a)^{(n+1)}/(n+1)!$$
$$S_n=f^{(n+1)}(C_q)(x-a)^{(n+1)}/(n+1)!$$

and where C_f and C_q are both between a and x. Clearly, we can cancel out a factor of $(x-a)$ and so, in the limit as x approaches a, we again obtain l'Hopital's Rule.

In conclusion, the author feels that the above ideas relating limits and l'Hopital's Rule to Taylor polynomial approximations is extremely beneficial to students. They will not only develop a better feel for limits, but also see some nice applications of other ideas of calculus, such as Taylor polynomials and the idea of approximation of one function by another. This reinforcement through repetition will serve them well in appreciating the power of approximation methods in other, more sophisticated contexts.

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References

[1] GORDON, S. P., and HUGHES HALETT, D., 1991, The implications of calculus reform on the two year colleges. *The AMATYC Review*, **12,** 50–57.
[2] STEMAN, P. W., 1991, Understanding limits. *Mathematics Teacher*, **84,** 286–287.
[3] GORDON, S. P., 1991, Introducing Taylor polynomial approximations in introductory calculus. *PRIMUS*, **1,** 305–313.
[4] MATHEWS, J. H., 1991, L'Hopital's Rule with Mathematica. *The AMATYC review*, **13,** 40–46.