LINKING MATHEMATICS AND PHYSICS VIA A CLASSROOM DEMONSTRATION

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ABSTRACT: The authors describe a simple cooling experiment that can be conducted in class at the college algebra, precalculus, calculus, or differential equations level. They utilize the notion of fitting functions to data to determine the best exponential function to fit the experimental data while reinforcing fundamental mathematical concepts. They then extend the approach to identify the underlying scientific principle, Newton's Law of Cooling, by solving either a difference equation or a differential equation.

KEYWORDS: Newton's Law of Cooling, Calculator Based Laboratory, behavior of functions, fitting functions to data.

INTRODUCTION

For much of the last century, mathematics and the various scientific disciplines have become increasingly isolated from one another. This has certainly been reflected in our curricula, where the applications taught in the mathematics courses tended to be somewhat artificial and where the other disciplines teach more and more of the mathematics they need.

In the last few years, things have started to change for a variety of reasons. Traditional discipline lines are becoming increasingly meaningless as current research and developments occur in fields that overlap two or more disciplines. This is reflected strongly in the scientific and engineering workplace, which is increasingly interdisciplinary in nature. The reform movement in mathematics has, among other things, placed a renewed emphasis on realistic applications to motivate the mathematical developments. Finally, modern technology now allows us to do things in the mathematics classroom that previously would have required moving an entire class over to a physics, chemistry or biology lab.

In the present article, we will illustrate how such a technology-oriented activity can be used in a mathematics classroom at the college algebra, precalculus, calculus, or differential equations level. Such an activity

- gives an experimental dimension to the mathematics,
- reinforces fundamental mathematical ideas and thinking,
- stresses the interplay between discrete and continuous perspectives,
- motivates the students to appreciate the value of the mathematics they are learning, and
- demonstrates the scientific method of using experiments to deduce physical laws.

Our experiment is based on Newton's Law of Cooling.

THE EXPERIMENT

We presuppose that students have been introduced to the notion of families of functions, whether in college algebra or a more advanced course. In particular, we presume that the students are sufficiently familiar with the behavioral characteristics of families such as linear, exponential, and power functions. We also presume that the students have been exposed to the idea of fitting functions to data, a powerful idea that has become fairly common because of the capabilities of graphing calculators and spreadsheets.

To run a simple experiment on temperature, all that is needed is a Calculator Based Laboratory (CBL) device with the temperature probe that comes with it, a graphing calculator with display device, two coffee mugs, and an immersion heater. In addition, a calculator program, such as the program Heat (or an equivalent one) that can be downloaded from the Texas Instruments website, `http://www.ti.com/calc/docs/downloads.htm`, is needed. (Comparable programs are available when a computer-based CBL unit is used.)

Fill one mug with cold water (a couple of ice cubes are fine, if available). Fill the other mug with warm water and heat it until the water just begins to simmer. Place the temperature probe into the heated water until it essentially reaches the temperature of the water. Then start the program HEAT. It will prompt you for the time interval between measurements; we suggest one-second intervals. Just as you are about to begin the data collection process, remove the probe from the heated water and immediately immerse it in the cool water. The program collects 36 temperature readings and displays the data graphically on the calculator. A typical set of data from such an experiment is shown in Figure 1; the associated table of temperature readings is shown in Table 1.

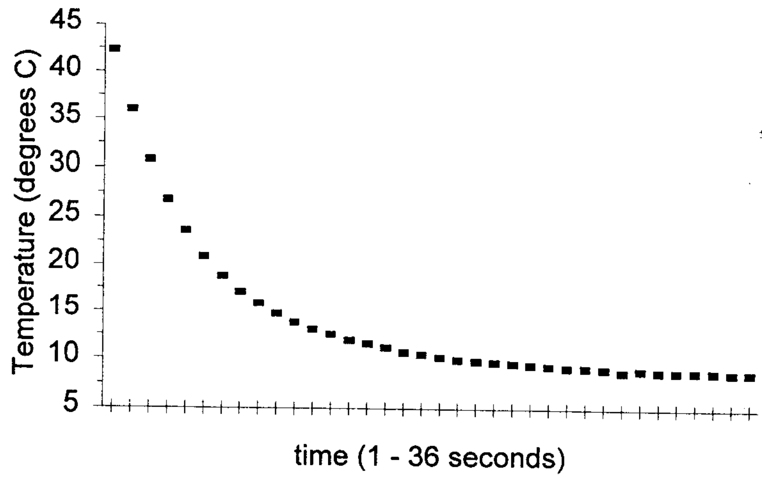

Figure 1. Scatterplot of experimental data on temperature readings.

Time	1	2	3	4	5	6	7	8	9	10	11	12
Temp	42.30	36.03	30.85	26.77	23.58	20.93	18.79	17.08	15.82	14.77	13.82	13.11
Time	13	14	15	16	17	18	19	20	21	22	23	24
Temp	12.51	11.91	11.54	11.17	10.67	10.42	10.17	9.92	9.80	9.67	9.54	9.42
Time	25	26	27	28	29	30	31	32	33	34	35	36
Temp	9.29	9.16	9.16	9.04	8.91	8.83	8.78	8.78	8.78	8.78	8.66	8.66

Table 1. Experimental Data – Temperature (°C) versus time.

THE MATHEMATICAL INVESTIGATION

Prior to conducting the experiment, it is effective to ask the students to predict the temperature pattern they expect to see. Does the temperature decrease linearly or, if not, does it decrease in a concave up or concave down manner? Once they are all convinced of the expected pattern, the experiment serves to reinforce their intuition.

Once the data has been collected and displayed, students should be asked to suggest possible candidates among the usual families of functions studied that behave in the manner shown to construct a continuous model to capture the trend in the discrete data points. Typical responses should be either a decaying exponential function or a decaying power function. With the calculator, it is quick and easy to try either or both of these possibilities, and both give reasonably good fits to the experimental data, though neither is an exceptional fit. However, the power function is not a good model for the process because it has a vertical asymptote at time $t = 0$.

At this point, we would lead the students to recognize that both functions decay to zero while the temperature readings decay to the temperature of the cold water. For our experimental data above, this is about 8.6°C.

The natural question to raise then is: How do we take this into account mathematically? With a little prodding, it is reasonable to expect that several students will suggest subtracting the 8.6 from each of the temperature readings to obtain a set of data that decays to zero. Having done this to the temperature data, the calculator then quickly provides the best-fit exponential function to the transformed data. For the above readings, this is

$$f(t) = 35.4394(0.8480)^t.$$

The graph of this function is shown superimposed over the transformed data in Figure 2 and we see that there appears to be an extremely close agreement. The corresponding correlation coefficient is $r = -0.9948$, which is exceptionally close to 1.

The question can then be posed about how we create a function that fits the original data and again it is reasonable to expect a number of students to suggest adding the 8.6 to the function $f(t)$ to create the final expression

$$T(t) = 8.6 + 35.4394(0.8480)^t.$$

This function is shown superimposed over the original temperature data in Figure 3. Along the way, there are many opportunities to reinforce ideas about shifting functions (including the tabular function) and the behavioral properties of the functions being used. It is also an especially good opportunity to point out the horizontal asymptote in a practical context. Finally, we note that the continuous model is an excellent fit to the discrete experimental data.

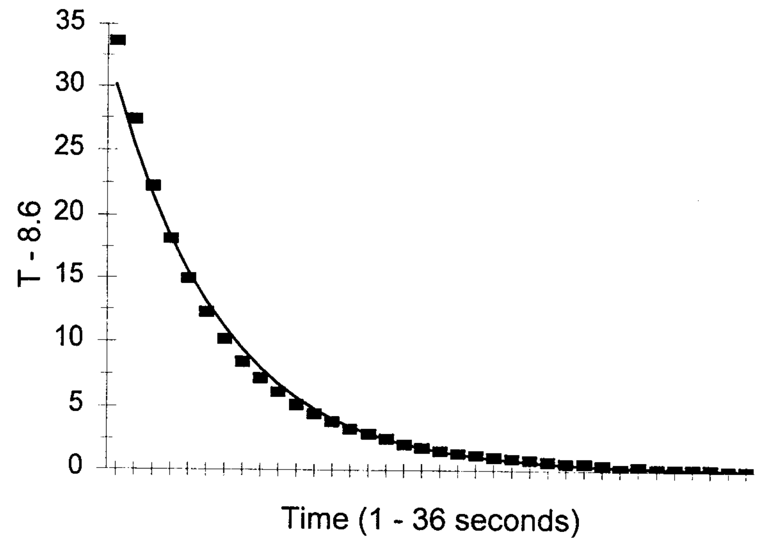

Figure 2. The exponential function that best fits the transformed data.

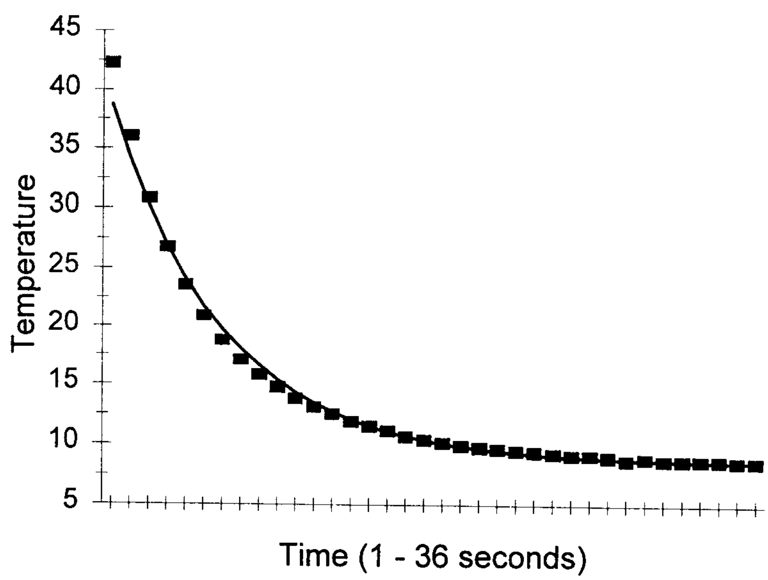

Figure 3. The shifted exponential function vs the experimental data.

We note that the entire process described can be done in about 15 to 20 minutes in class, so it does not entail losing a significant amount of class time. On the other hand, the gain in student motivation and understanding can certainly be significant.

THE UNDERLYING SCIENTIFIC PRINCIPLE

The above mathematical analysis is limited to merely observational and experimental activity. However, the hallmark of the sciences is developing an understanding of the actual process and identifying an underlying scientific principle. With this in mind, let's reconsider the experimental data.

In the process of learning about the characteristics of linear functions, we presume that students are familiar with the fact that a linear function is characterized by constant differences between successive terms for a fixed change in the independent variable. We can build on this by pointing out that considerable information can be obtained about a process by looking at the differences in the data readings. To do this, we use a simple calculator program (given in the Appendix) to calculate the successive differences for the experimental temperature readings. (Note that the CBL's HEAT program stores the time readings in List 3 and the temperature readings in List 4 for the TI-82 or TI-83 calculators. We use List 5 for the shifted temperature values, L4 - 8.6. The program DELTA then stores the successive differences in List 6.) If we look at these differences numerically or look at a plot of the differences, ΔT, versus the time readings, t, as shown in Figure 4, it is evident that ΔT is not a linear function of t.

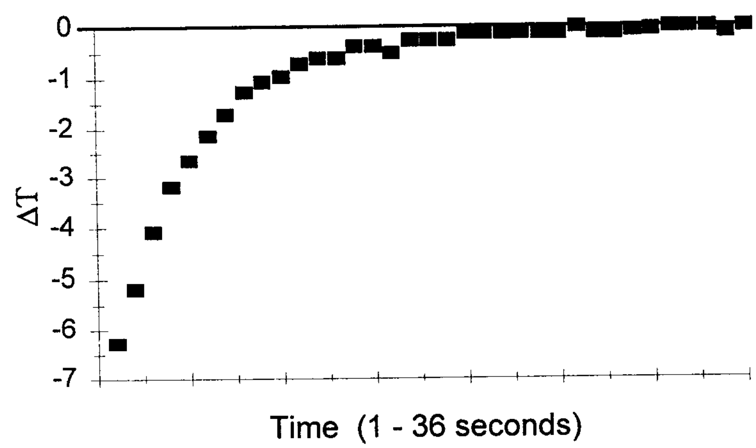

Figure 4. Scatterplot of successive differences of temperature readings vs time t.

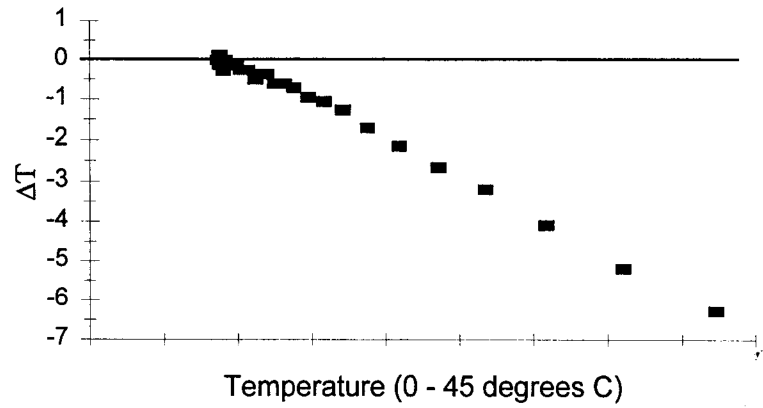

Figure 5. Scatterplot of successive differences of temperature readings vs temperature T.

However, a very different picture emerges if we plot ΔT versus T, as shown in Figure 5. From this, it is clear that ΔT appears to be a linear function of T. Incidentally, note that the isolated points at the lower right corner of Figure 5 correspond to the earlier high temperatures where there is a large difference in temperature values; as the temperature drops, the differences between them decrease and the corresponding points are ever more tightly clustered in the upper left corner of the scatterplot. Using the calculator, we find that the best linear fit to these data values is

$$\Delta T = -0.1851T + 1.6740.$$

The corresponding correlation coefficient is $r = -0.9976$, which indicates an almost perfect linear fit.

If we now factor out the coefficient of T, -0.1851, we obtain

$$\Delta T = -0.1851(T - 9.04).$$

Notice that the value -9.04 that occurs is fairly close to the temperature of the cool water. (To be honest, the authors expected that it would be closer, though that could be attributable to errors in measurement.) In other words, the change in temperature ΔT is proportional to the difference between the temperature T of the probe and the temperature of the medium, 8.6. It is precisely this observation that is the basis of Newton's Law of Cooling.

We can actually carry this analysis a step further. First, we consider what can be done at a precalculus level where the students have been exposed to some simple ideas about difference equations. The expression

$$\Delta T_n = T_{n+1} - T_n = -0.1851T_n + 1.6740,$$

which is equivalent to

$$T_{n+1} = T_n - 0.1851T_n + 1.6740 = 0.8149T_n + 1.6740,$$

is a relatively simple difference equation. The general solution to the difference equation

$$x_{n+1} = ax_n + b$$

is given by

$$x_n = Ca^n + \frac{b}{1-a},$$

where C is an arbitrary constant. See, for example, [1, 2, or 3]. For our coefficients, we have

$$\frac{b}{1-a} = \frac{1.6740}{0.1849} = 9.04.$$

Thus, the general solution for our difference equation is

$$T_n = C(0.8149)^n + 9.04.$$

If we impose the initial condition given by the initial temperature of the probe, $T_0 = 42.3$, we obtain the specific solution

$$T_n = 33.26(0.8149)^n + 9.04, \quad (1)$$

which is quite close to the function $T(t) = 35.4394(0.8480)^t + 8.6$ that we obtained above by fitting the exponential function to the data.

Alternatively, at the calculus or differential equations level, we might reason that if the time intervals between measurements is fairly small, then the differences are good estimates of the derivative of the temperature function and so we have the simple differential equation

$$T' = -0.1849T + 1.6740 = -0.1849(T - 9.04).$$

When we integrate this equation, we obtain

$$T = 9.04 + Ce^{-0.1849t}$$

where C is a constant of integration. When we use the same initial condition here, we find that

$$\begin{aligned} T &= 9.04 + 33.26e^{-0.1849t} \\ &= 9.04 + 33.26(0.8312)^t. \end{aligned} \quad (2)$$

The discrepancy between the solution to the difference equation (1) and the solution to the differential equation (2) is attributable to the fact that we used the forward differences, $\Delta T = T_{n+1} - T_n$, as estimates for the derivative T'. The use of central differences instead of forward differences would give better approximations to the derivative T' and so would likely improve the accuracy of the estimates.

ACKNOWLEDGMENT

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APPENDIX - CALCULATOR PROGRAM DELTA

```
: 1→I
: While  I < 36
: L4(I + 1) − L4(I)→L6(I)
: I + 1→I
: End
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BIOGRAPHICAL SKETCHES

Sheldon P. Gordon is professor of mathematics at SUNY College of Technology at Farmingdale. He was project director of the NSF-supported *Math Modeling/PreCalculus Reform Project* and principal author of the project text, *Functioning in the Real World: A PreCalculus Experience*. The project was recently awarded the top prize in the algebra through precalculus category in a competition sponsored by the Annenberg Foundation/Corporation for Public Broadcasting to identify exemplary Innovative Projects Using Technology. Gordon is a member of the Harvard Calculus Consortium that has produced the texts *Calculus*, *Multivariable Calculus*, *Applied Calculus*, and *Brief Calculus*. He is the co-editor of the MAA Notes volumes, *Statistics for the Twenty First Century* and *Calculus: The Dynamics of Change*, and is the author of over 90 articles.

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