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LINKING THE SINE AND COSINE FUNCTIONS TO POLYNOMIALS AT THE PRECALCULUS LEVEL

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One of the most important concepts in mathematics is the idea of approximating one function by another. In the present article, we show how this concept can be introduced in a variety of ways in which the sine and cosine functions can be constructed, that is approximated, using polynomials. Each of the methods used involves central ideas that are currently being introduced at the precalculus level, as in [1], that highlight the interplay of graphical, numerical, and symbolic representations of functions and show the connections between different aspects of mathematics, particularly the behavior of polynomials, the behavior of the sine and cosine, and the use of trig identities to link them.

We begin with an examination of the graph of the sine function near the origin, which clearly indicates a linear pattern. (Just zoom in sufficiently on the curve near 0 with a graphing calculator.) Therefore, near the origin, the sine curve can be well-approximated by that line. An analysis of numerical values of the sine function near 0 suggests an approximate slope of 1 for that line, so it is reasonable to write, when x is close to 0, $\sin x \approx x$. Of course, if you zoom out a bit, the graph of the sine function eventually bends away from the line, so the approximation is only effective when x is very close to the origin, say between $x = -.6$ and $x = .6$. The problem we face is: How can we improve on this approximation so that we have a function that better fits the bends of the sine curve?

Constructing the sine and cosine using curve fitting

One of the most powerful ideas in mathematics is the notion of constructing functions to fit a set of data. Suppose you examine the graph of the sine function on a slightly larger interval, say from -1.6 to 1.6. If you didn't know anything about the trig functions, you might conclude that the curve really looks like a cubic. Let's try to construct the equation for an appropriate cubic that closely fits this portion of the sine curve. To do so, we use the standard values for the sine function,

x	$-\pi/2$	$-\pi/3$	$-\pi/4$	$-\pi/6$	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
$\sin x$	-1	-.866	-.707	-.5	0	.5	.707	.866	1

Using these values as a set of data with a calculator's cubic curve-fitting routine, we obtain

$$\sin x \approx S_3(x) = -.142122x^3 + .986700x.$$

We show the graph of this cubic compared to the sine function on the interval from -2.5 to 2.5 in Figure 1. It is virtually impossible to distinguish by eye between the two curves between $-\pi/2$ and $\pi/2$ (unless you zoom in considerably). For a feel for the numerical accuracy of this approximation, consider the following table. (The comparable values occur for negative values of x because of symmetry of both the sine function and the cubic, which consists of odd power terms only.)

x	0	.2	.4	.6	.8	1	1.2	1.4	1.6
$\sin x$	0	.1987	.3894	.5646	.7174	.8415	.9320	.9854	.9996
$S_3(x)$	0	.1962	.3856	.5613	.7166	.8446	.9385	.9914	.9966

We see that this cubic polynomial gives roughly two decimal accuracy across this interval.

Let's now see how well this cubic approximation to the sine stacks up with the cubic Taylor polynomial approximation $T_3(x) = x - .166667x^3$. The coefficients of both cubic terms are quite close and so the approximations based on either should be reasonably close also. You may want to compare both polynomials and the sine function graphically or numerically to see how they relate to each other.

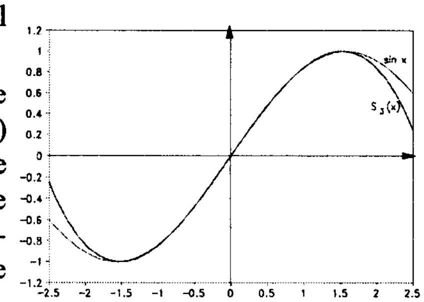

Figure1

What can we do to approximate the sine function beyond $\pi/2$? Clearly, the cubic polynomial bends away from the sine curve in both directions, so we cannot use it, directly, to estimate the values of sin x for larger values of x. However, we can use some fundamental properties of the sine function to circumvent this problem. First, the periodicity of the sine tells us that any angle greater than 2π can be reduced to an equivalent angle between 0 and 2π, so all we need is a reasonably accurate approximation to sin x between 0 and 2π. Second, the graph of sin x between π and 2π is precisely the reverse mirror image of the graph between 0 and π, so we actually only need to approximate the sine function accurately between 0 and π. Third, the first arch of the sine curve is symmetric about $x = \pi/2$, so in fact our reasonably accurate approximation, $S_3(x)$, which gives two decimal accuracy between 0 and $\pi/2$, actually suffices to estimate values for the sine function for any choice of x, using some judicious reasoning.

What if we want more than two decimal accuracy? Unfortunately, if all we have at our disposal is a graphing calculator that only (!) provides the ability to fit polynomials up to fourth degree to a set of data, we have no way to improve on the level of accuracy using this approach. However, there are other approaches that we will consider below.

Before going on, though, let's consider a comparable analysis with the cosine function. When you look at the cosine between -1.8 and 1.8, say, it suggests a parabolic shape; on a somewhat larger interval, say from -4 to 4, the shape is more suggestive of a fourth degree polynomial. So let's apply the above analysis to fit both a quadratic and a quartic to a set of data based on the standard angles:

x	$-\pi/2$	$-\pi/3$	$-\pi/4$	$-\pi/6$	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
cos x	0	.5	.707	.866	1	.866	.707	.5	0

Using a calculator's curve-fitting routine, the best-fit quadratic function for this set of data is

$$\cos x \approx C_2(x) = -.397356x^2 + .965192,$$

since the coefficient of the linear term is virtually zero. We show the graph of this polynomial compared to the cosine curve on the interval from -2 to 2 in Figure 2. The two curves appear reasonably close from about $-\pi/2$ to $\pi/2$. Alternatively, we can fit a quartic to this set of data and so find that the best-fit fourth degree polynomial is

$$\cos x \approx C_4(x) = .037005x^4 - .496385x^2 + .999531,$$

since the coefficients of the odd power terms are effectively zero. Of course, the argument we used above about reducing any value of x to a corresponding one between 0 and $\pi/2$ applies to the cosine as well.

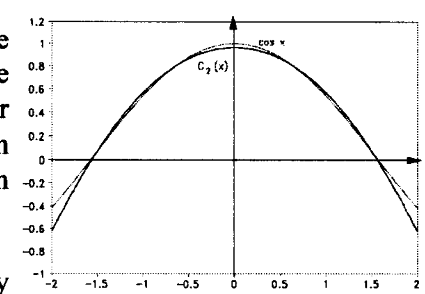

Figure2

In the table below, we show how well the two different approximations agree with the actual values for the cosine function at a variety of points. Notice that the quadratic approximation only gives about one decimal accuracy across the interval. Of course, if you extend it beyond $\pi/2$, the level of accuracy drops rapidly. However, the fourth degree polynomial certainly agrees considerably better with the cosine. From 0 to $\pi/2$, they agree to almost three decimal places. Beyond that, up until about 2.1, the agreement is reasonably good, though thereafter the quartic also diverges from the cosine function.

x	0	.2	.4	.6	.8	1.0	1.2	1.4	1.6
$\cos x$	1	.9801	.9211	.8253	.6967	.5403	.3624	.1700	-.0292
$C_2(x)$	.9652	.9493	.9016	.8221	.7109	.5678	.3930	.1864	-.0520
$C_4(x)$	.9995	.9793	.9211	.8256	.6970	.5402	.3615	.1688	-.0287

Constructing Better Approximations using the Errors

Unfortunately, as we pointed out before, because calculators are typically limited to a maximum of a quartic fit to a set of data, we cannot continue this approach to obtain better approximations to the sine or cosine. Instead, we will approach this same problem from a different point of view by examining the behavior of the errors -- the differences between a function and its approximation. Thus, if we start with $\sin x \approx x$, the corresponding error function is

$$E_1(x) = \sin x - x.$$

We show the graph of this function on the interval [-1,1] for x and [-.1,.1] for y in Figure 3. First, we see that the values for this error function are quite small most of the way across the interval, so that the linear approximation to the sine function is quite good. Second, notice that the behavior of the error function strongly suggests a cubic function with a negative leading coefficient. In fact, the manner in which the error function passes through the origin suggests a pure cubic -- that is, a multiple of x^3. To find an equation for this cubic function, we create a set of points for the error function and find the cubic that best fits them. But, this requires a little care because fitting a power function to a set of data pairs (x, y) actually involves fitting a line to the transformed data set $(\log x, \log y)$; this requires that all the original data values be positive. For instance, if we were to select the angles 0°, 15°, 30°, ..., 90°, then we would obtain the table of values with x in radians:

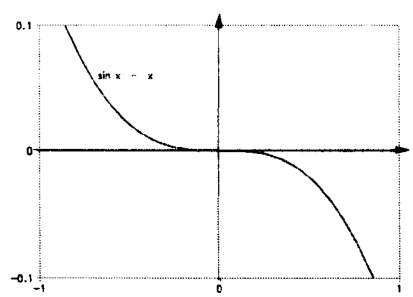

Figure3

x	0	.2618	.5236	.7854	1.0472	1.3090	1.5708
$E_1(x)$	0	-.00298	-.02360	-.07828	-.18117	-.34307	-.57080

Unfortunately, the values for the function $E_1(x)$ are zero or negative and so we cannot use the above entries as data for fitting a power function. Alternatively, if we tried to use positive values for the function $E_1(x)$, then the corresponding values of x would be negative. Therefore, we make several slight adjustments in the table of values. For one, we avoid $x = 0$ by considering $x = .0001$ instead. Also, we consider $-E_1(x)$ instead corresponding to the other values of x. We thus will work with the table of values

x	.0001	.2618	.5236	.7854	1.0472	1.3090	1.5708
$-E_1(x)$	1.66666×10^{-13}	.00298	.02360	.07828	.18117	.34307	.57080

with the understanding that we will readjust after performing the regression analysis. (If we use $|E_1(x)|$ instead, then we lose the graphical image of the power function with an odd integer power.) The best-fit power function to this set of data is

$$y = .157566x^{2.99245},$$

with a corresponding correlation coefficient of $r = .999993$. When we undo the negative multiple we introduced, this essentially tells us that

$$\sin x \approx x - .157566x^3.$$

compared to the cubic Taylor polynomial approximation

$$\sin x \approx T_3(x) = x - x^3/6 = x - .166667x^3.$$

If we performed the same analysis using a set of x-values that were much closer to the origin, say $x = .1, .2, ..., .5$, then we would have obtained even more accurate results $y = .164212x^{2.99306}$.

We now extend the above line of reasoning to improve on the approximation. Consider the error function

$$E_3(x) = \sin x - (x - x^3/6),$$

shown in Figure 4 for x in [-1,1] and for y in [-.01,.01]. Visually, we see that the values for this function are quite small across most of the interval, so the cubic approximation is extremely accurate. We also see that the curve behaves like a polynomial of odd degree, being very flat as it crosses the x-axis. This suggests that, near the origin, we might use a fifth polynomial with positive leading coefficient. If we use the values

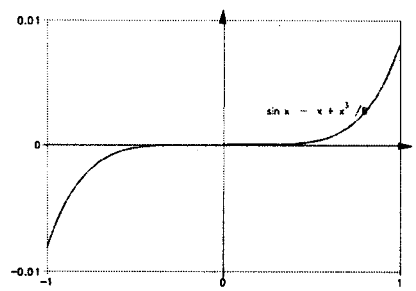

Figure4

x	.01	.02	.03	.04	.05
$E_3(x)$	8.334×10^{-13}	2.666×10^{-11}	2.2049×10^{-10}	8.5330×10^{-10}	2.6040×10^{-9}

then the resulting best-fit power function is

$$y = .0083314x^{4.999942} \approx x^5/120,$$

with a correlation coefficient given as $r = 1$, due to rounding. Consequently,

$$E_3(x) = \sin x \approx x - x^3/6 + x^5/120.$$

(If we use the error based on the cubic approximation $\sin x \approx x - .157566x^3$, we would obtain a somewhat worse value.) We can obviously continue this process to achieve still better approximations to the sine function, though care must be taken to avoid negative values for either x or y.

The identical type of analysis can be carried out to produce successively better approximations to the cosine function as well. However, we will not go into this here at all.

Constructing sine and cosine approximations using trig identities

We next approach the question of constructing the sine and cosine functions with a different method that is based on the use of some standard trig identities. These methods can be used both to reinforce students' facility with the identities and to provide some motivation for their use. We start with our linear approximation to the sine function, $\sin x \approx x$, and use it conjunction with one of the double angle identities for the cosine,

$$\cos x = 1 - 2\sin^2(\tfrac{1}{2}x) \approx 1 - 2(\tfrac{1}{2}x)^2 = 1 - x^2/2.$$

Thus, we obtain a quadratic approximation to the cosine function near the origin which is, in fact, the quadratic Taylor polynomial for the cosine.

We now use the double angle identity for the sine along with the above two approximations to obtain

$\sin x = 2\sin(\tfrac{1}{2}x)\cos(\tfrac{1}{2}x) \approx 2(\tfrac{1}{2}x)[1 - (\tfrac{1}{2}x)^2/2] = x - x^3/8.$

While not precisely equivalent to the Taylor polynomial of degree 3, this is still quite a good approximation to the sine curve, as shown in Figure 5 where we display the three graphs: $y = \sin x$, $y = x - x^3/8$, and the Taylor polynomial $y = x - x^3/3!$. In fact, without the labeling on the graphs, it is not instantly apparent which curve is which. Further, observe that while the Taylor polynomial remains close to the sine curve over a somewhat longer interval centered at the origin, the non-Taylor curve follows the bends in the sine curve somewhat better.

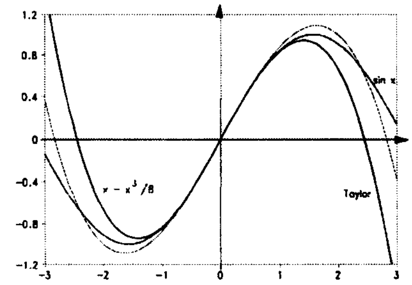

Figure5

We continue this process using

$$\cos x = 1 - 2\sin^2(\tfrac{1}{2}x) \approx 1 - 2[(\tfrac{1}{2}x) - (\tfrac{1}{2}x)^3/8]^2$$
$$= 1 - x^2/2 + x^4/2^5 - x^6/2^{11}.$$

Observe that this polynomial consists of only even powers of x, which makes sense because the cosine function is even and any functions that are used to construct it should also be even functions. Moreover, although not quite the same as the sixth degree Taylor approximation,

$$\cos x \approx 1 - x^2/2 + x^4/4! - x^6/6!,$$

this does represent a very good approximation to the cosine over a larger interval centered at the origin. We show this in Figure 6. As before, observe how well the non-Taylor polynomial follows the behavior of the cosine curve and is a far better fit to the cosine over a considerably larger interval than the quadratic approximation shown in Figure 2.

Interestingly enough, we obtain quite different, albeit still quite accurate, approximations to the cosine function if we use the other two versions of the double angle formula for the cosine:

$$\cos x = 2\cos^2(\tfrac{1}{2}x) - 1 \approx 1 - x^2/2 + x^4/2^5$$
$$\cos x = \cos^2(\tfrac{1}{2}x) - \sin^2(\tfrac{1}{2}x) \approx 1 - x^2/2 + x^4/2^5 - x^6/2^{12}.$$

The process can clearly be continued, though the algebra quickly becomes more than one should ask a student to perform. Using a computer algebra system such as Derive, we obtain

$$\sin x \approx x - 5x^3/2^5 + 3x^5/2^9 - 9x^7/2^{17} + x^9/2^{22}$$
$$\cos x \approx 1 - x^2/2 + 5x^4/2^7 - 37x^6/2^{15} + 129x^8/2^{23} - 59x^{10}/2^{29}$$
$$+ 59x^{12}/2^{37} - 129x^{14}/2^{47} + 9x^{16}/2^{53} - x^{18}/2^{61}$$

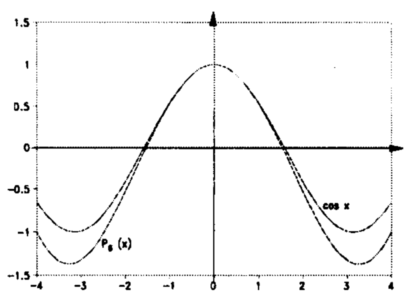

Figure6

Notice that all of the terms in the cosine approximations involve even powers, while those in the sine approximations involve odd powers. Also, the numerical values for the coefficients are quite close to those in the Taylor approximations, at least for the first few terms which dominate near the origin. Consequently, these expressions do provide quite good approximations to the sine and cosine.

Unfortunately, each successive approximation does not merely extend the previous one by using the same terms, as is the case with the successive Taylor approximations, nor does there appear to be any obvious patterns in the coefficients obtained. Further, since each successive approximation is based on using a series of prior estimates, we should expect that the errors in the approximations will continually mount up, so that the differences in accuracy between Taylor polynomials and these non-Taylor polynomials will increase as the degrees increase.

Reference

1. Gordon, Sheldon P., Florence S. Gordon, B. A. Fusaro, Martha Siegel, and Alan Tucker, *Functioning in the Real World: A PreCalculus Experience*, Addison-Wesley, 1997.