The State of Mathematics Education Today:
What Happens in the Classroom

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When academics outside of mathematics think about mathematics education, they invariably picture the courses and approaches that they went through as students. This applies even in disciplines such as modern economics that use mathematics very heavily. It even applies in disciplines whose use of mathematics is quite different from what those faculty members were exposed to as students and, somewhat surprisingly, despite calls from people in their own field for changes in the mathematical preparation of students coming into their courses.

In this article, I will attempt to provide an overview of what the current state of the mathematics curriculum is and how it may (or may not) support the mathematical needs of students in economics. In some ways, very significant changes have taken place in mathematics education over the past 20 years; in others, virtually no change has taken place. The actual status is very much a function of the mathematics department in a particular institution, if not the individual members of that department.

About 20 years ago, some mathematicians became concerned about the state of calculus education. They saw that the existing courses were very unsuccessful, with success rates typically of 50% or less. Students who successfully completed a calculus course often experienced considerable difficulty in the subsequent course, if they took any subsequent math course at all. For decades, mathematicians had observed a continual decline in the algebraic ability of students, so that the focus in calculus had grown increasingly on improving algebraic skills by spending more and more time on routine problems involving the rules of differentiation and integration. (An old adage in mathematics says: "You take calculus to learn algebra." A follow-up adage says: "You take differential equations to learn calculus.") In turn, the applications of calculus became more and more focused on template problems whose solutions many students would memorize without understand and regurgitate on tests and less and less on applications and concepts that students might be able to carry over to their courses in other disciplines.

At the same time, complaints about calculus instruction from several of the partner disciplines, most notably engineering and physics, became quite strident. Students were not able to transfer knowledge from their calculus classes to applications in those other fields – for instance, they did not seem to understand what the derivative of a function tells you about the function. Far too much class time in these other fields was being spent (wasted?) in teaching the key mathematical concepts needed and ways to apply those concepts.

What emerged was a national effort to reform calculus instruction. The guiding principles behind this movement included:

1. Putting considerably more emphasis on conceptual understanding through non-routine examples and problems and realistic problem-solving through mathematical modeling.

2. Reducing the emphasis on the development of algebraic skills.
3. Emphasizing multiple representations of functions and other concepts – looking at them graphically, numerically, algebraically, and verbally/dynamically.
4. Increasing the emphasis on differential equations as dynamic models.
5. Covering less material, but doing it in considerably greater depth.
6. Engaging the students in learning and doing calculus, not just being passive recipients of information. This involved a variety of strategies, including group learning, individual and group projects involving preparing formal reports, enhancing students' communication skills, using portfolios and other alternative assessment methods, and so on.
7. Integrating the routine use of technology (typically graphing calculators) throughout calculus as both a teaching and a learning tool.

To support this effort, the NSF developed a major initiative that funded a relatively small number of large-scale projects intended to develop full sets of calculus materials that would have a national impact and a large number of relatively small projects intended to implement the major projects by adapting them to local settings.

In turn, the efforts to reform calculus quickly led to comparable efforts to affect other mathematics offerings, including post-calculus courses such as differential equations and linear algebra, and courses below calculus such as precalculus and college algebra.

A great many individuals and departments bought into this reform movement and most reported considerable success. Many other mathematicians felt that the reform effort was somehow undermining the spirit of mathematics and fought against it bitterly, leading to what have been called "math wars".

Long Term Impact - The long-term effects of this reform movement are hard to assess. The kinds of conceptual problems that typified the calculus reform textbooks have been picked up and incorporated into virtually all of the "traditional" textbooks. So, one can certainly conclude that the initiative has shifted the entire calculus enterprise in the direction of reform. In the process, however, all but one of the reform textbooks that grew out of the movement have been dropped by the commercial publishers, whose focus has been increasingly on publishing only blockbuster textbooks that would appeal to *all* potential adopters.

Unfortunately, the new material incorporated into these books tends to be add-ons at the end of sections or the end of problem sets and few people actually have the time or inclination to get to them. They don't fit easily into a course that is may be focused on developing algebraic skills and covering every possible traditional topic. So, one could also argue that, like the traditional land war in Asia, every invading army has been swallowed up by the sheer size of the enterprise.

One significant aspect of this has to do with the use of technology. Certainly, in high school, the use of graphing calculators for mathematics education has become ubiquitous. A student literally cannot hope to do well on any of the high-stakes tests, such as the SAT or the Advanced Placement test in calculus, without using a graphing calculator. Many schools now integrate their use as early as junior high school, so almost all college-bound students have used them for years throughout their mathematics education. The various college-level mathematics professional societies conduct an extensive survey to ascertain the state of mathematics education every five years. In the most recent (Lutzer, et. al., 2007) survey, it was found that only about 50% of all students in college calculus are allowed to use technology. But, without technology, most of the other reform efforts become impossible to implement, so the implication is that the reform movement still has a long way to go.

In a related direction, as has been reported in the introductory article in this Symposium, the MAA's committee on Curriculum Renewal Across the First Two Years (CRAFTY) conducted the first round of its Curriculum Foundations project several years ago. In this project, leading educators from 17 disciplines, primarily those that traditionally have been considered highly quantitative, were brought together to discuss and prepare recommendations to the mathematics community on the mathematical needs of their students today. In reading the recommendations (Ganter and Barker, 2004), it is amazing to see the degree of uniformity that exists about those needs. In particular, virtually every discipline recommends:

- There should be a very strong emphasis on problem solving.
- There should be a very strong emphasis on mathematical modeling.
- Conceptual understanding is more important than skill development.
- The development of critical thinking and reasoning skills is essential.
- The use of technology, especially spreadsheets, is extremely important.
- A major emphasis should be on the development of communication skills (both written and oral).
- There should be a much greater emphasis on probability and statistics.
- There should be greater cooperation between mathematics and the other fields.

Furthermore, these are precisely the same sentiments expressed by the economists who participated in the Curriculum Foundations project workshop on Mathematics and Economists. (Very similar recommendations also came out of the other workshops in the second round of the CF project.)

Despite this convergence of opinion among most of the disciplines that send us students, the large majority of mathematics departments has not particularly listened to or, at least, has not taken the recommendations to heart. Most are still offering courses with the same philosophy (primarily preparing students for the next math course) that they have had for decades. In large measure, this can probably be ascribed to doing what is familiar and comfortable. It can also likely be attributed to the fact that most mathematicians are not all that familiar with the recommendations that have been coming out of the CF project or perhaps to a belief that those recommendations don't apply in their institution since the other departments on campus are not making a case for change locally. (Sometimes, though, the issue can be one of words having different meanings in different disciplines. For instance, many physicists and engineers have criticized mathematics courses as being too theoretical. Mathematicians interpret this as meaning that we spend too much time on theorems and proofs, but the practitioners really mean that the focus in the math classes is primarily on contrived, non-contextual problems rather than real-world applications. In any dialog between departments, very specific examples of problems are essential; certainly this is much more likely to be effective than phrases that can be interpreted differently.)

As a matter of fact, CRAFTY's picture of how the CF reports can best be utilized locally is for the other disciplines to use them as a vehicle for opening discussions with the mathematics department. We suspect that that would be a far more effective approach than expecting some members of the mathematics department to lobby the rest of the department to make such changes or even to approach the other departments.

Other Mathematics Offerings - As mentioned above, the calculus reform movement has carried over to other mathematics offerings. Two post-calculus courses in particular have undergone very extensive changes and those changes appear to have been adopted very widely. One of these is the introductory course in differential equations, which used to be focused on solving differential equations in closed form with all manner of classical (manipulative) techniques. The focus now is far more on differential equations as models for phenomena from all areas, not just the physical sciences. Technology is used routinely to create graphical images of the solutions, which are approximated by numerical methods, and students focus on the quantitative behavior of the solutions and how well they agree with the realistic phenomena that are being modeled. Clearly, this course is now very much in the spirit of the principles underlying calculus reform as well as the recommendations from the CF project workshops.

The other post-calculus course that has changed dramatically is the first course in linear algebra, though the changes have not been adopted quite so universally. In the past, this course was offered as if the only students taking it were math majors and it was extremely theoretical with an overwhelming emphasis on theorem and proof in support of the theory of abstract vector spaces. Over the past two decades, there has been explosive expansion of enrollment of students from other disciplines, including the social sciences (certainly economics), computer sciences, and engineering. These students, for the most part, do not need the emphasis on vector spaces, but rather on the theory of matrices and their applications in many other fields. At a large number of schools (likely a majority), the linear algebra course has adapted to reflect this. In fact, many high schools now incorporate some exposure to matrix algebra and its applications (say in solving systems of linear equations) as part of the regular curriculum.

The calculus reform movement has also impacted the courses below calculus, especially precalculus, though the impact has not been extremely widespread. A number of projects were funded by NSF with the express intent of carrying the same reform philosophy over to these offerings, so that there would be a heavy focus on conceptual understanding and problem solving via mathematical modeling, as well as the routine use of technology for both the teaching and learning of mathematics. Another major emphasis has been the inclusion of significant amounts of real-world data and the use of data fitting routines on the calculator or software to create models based on most of the standard classes of functions (linear, exponential, logarithmic, power, polynomial, and sinusoidal) normally treated in these courses. Moreover, CRAFTY has developed and published an official set of recommendations calling for *all* college algebra courses to be offered in the same spirit. As outgrowths of those efforts, virtually all of the mainstream precalculus and college algebra texts have incorporated examples and problems reflecting these themes. Unfortunately, little if anything has been removed from these books to make the time or room for the newer topics to be covered. The result is that, as with the mainstream calculus texts, the ideas and problems are there, but are far too often overlooked by instructors who are pressed to finish a more traditional syllabus.

Most mathematics departments do offer another course that should be of interest to economics faculty, a calculus course that is designed for non-science students. This is usually a one-semester offering, though occasionally it is a two-semester sequence. The intended audience is students in business and other social sciences and possibly those in the biological sciences. Typically, from a mathematician's viewpoint, this is a "watered down" version of calculus in which the harder algebraic topics and methods have been eliminated, and less attention is paid to the underlying theory. In the process of watering it down, the course often focuses almost exclusively on polynomial calculus, because the rules for dealing with polynomials seem much simpler than those for other classes of functions. In the process, there may be little attention paid to exponential and logarithmic functions, even though those functions are far more important in the social and biological sciences than polynomials are. Furthermore, things like any discussion of parameters tend to be avoided because they are considered too sophisticated for the audience. The applications and problems tend to be very artificial situations that have a veneer of being from the subject areas, but in reality are not at all indicative of how calculus would actually be applied in those fields. The use of real data is almost non-existent in these courses (as, in fact, in virtually all courses offered by math departments with the exception of introductory statistics), because working with tables is unfamiliar territory to most mathematicians.

From an economist's perspective, the reality is that at least 95% of mathematicians cannot distinguish between economics and business. That's why economics was not part of the first round of the Curriculum Foundations project (though business was) – we simply didn't know that there was a difference! In fact, the typical mathematician likely knows less economics than the freshmen in your introductory *Principles* courses. This is something that is vital for economists to understand in any contacts with the mathematics faculty.

If you want to get a fairly accurate assessment of the philosophy in your mathematics department, perhaps the easiest way to do it is to walk over to the bookstore and leaf through the texts for the lower division math courses. In particular, glance at a representative sample of the problem sets. You can easily recognize a course whose focus is primarily on algebraic manipulation (and not on concepts and applications) if what you see is predominately large collections of problems with instructions to do something to each of the following: like factor, expand, solve, differentiate, integrate, etc., and where some 90% of those problems use x as the unknown or variable. (Most mathematicians have not learned that x is rarely the variable of choice in other disciplines; in fact, this author believes that the almost exclusive use of *x* and *y* is one of the major reasons that most students have trouble transferring mathematical knowledge to other disciplines.)

On the other hand, if you see lots of graphs and tables, then there is a reasonably good chance that the philosophy for the course is one that is supportive of, or at least open to, faculty in other disciplines. The graphs and tables are a hallmark of an emphasis on conceptual understanding and the tables typically highlight the fact that the book focuses on real-world applications.

In this context, think about finding the equation of a line. In traditional mathematics courses, students are repeatedly asked to do it, and this begins in pre-algebra courses, in introductory algebra, in intermediate algebra, in college algebra, and in precalculus. Yet, many students still have trouble finding the equation of the tangent line to a curve in calculus. And, from the point of view of other disciplines, particularly those that are data-driven or have laboratory components, most students seem never to have learned how to find a line. The difference comes down to the way that this is presented in math. The usual problem is something like: Find the equation of the line through the points (1,5) and (3, 9). The two points invariably have integer coordinates, usually only one digit; the variables are invariably *x* and *y;* and the worst that the slope comes out to be is :. In comparison, in most other disciplines, the student is presented with a set of data (often from a lab experiment), needs to draw a scatterplot and a line that captures the pattern, then has to find the equation of a line that fits the data, and finally to use that line to answer predictive questions in context. Moreover, the variables are typically letters that reflect the context. It is little wonder that most students cannot transfer the overly simplistic ideas from their math classes to such a context.

References

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