Preparing Students Mathematically for the Twenty First Century

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Background Major changes have taken place in the mathematical education of students in the United States over the last decade. These changes have come about for a variety of reasons, including (1) the growth of technology and what it can provide for the teaching and learning of mathematics, (2) the changing mathematical needs among the people who use mathematics, and (3) the changing demographics of the students taking mathematics.

Year	1955	1965	1970	1975	1980	1985	1990	1995
Enrollment	2.66	5.92	8.58	11.19	12.10	12.25	13.82	14.95

Table 1

Table 1 shows the annual collegiate enrollment, in millions, in the United States since 1955. Since the pattern in enrollment is actually concave down, an appropriate choice to fit the data is a power function of the form $y = A\,t^p$, with $0 < p < 1$. Using the regression features available on all graphing calculators, a power function that fits this data well is $D(t) = 0.7345\,t^{0.8053}$, where $t = 0$ corresponds to the year 1950. This function is the solid curve in Figure 1, which is superimposed over the data points. The corresponding correlation coefficient, $r = 0.9917$, indicates a very high level of correlation.

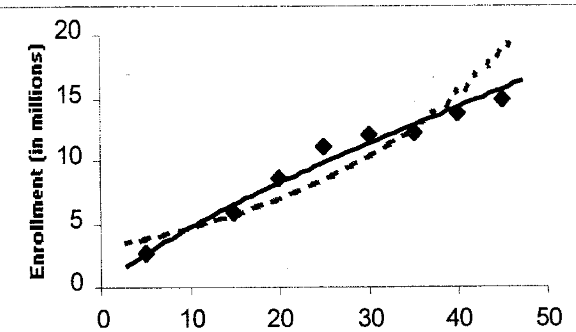

Figure 1

In recent years, it has become common to hear the media describe any situation involving rapid growth as exponential, whether or not that is an appropriate model. Thus, for comparison, note that the exponential function that best fits this data over the same time period is $C(t) = 0.421(1.0408)^t$, where $t = 0$ also corresponds to the year 1950. The base, or growth factor, of 1.0408 indicates that collegiate enrollment has been growing at an annual rate of about 4% over this time period. The associated correlation coefficient, $r = 0.9234$, indicates a high degree of correlation although it is not a high as the correlation associated with the power function. This function (the dotted curve) is shown superimposed over the data values in Figure 1 and we see that it is not a particularly good fit because of the concave down pattern in the data.

Year	1960	1965	1970	1975	1980	1985	1990	1995
Math Enroll.	0.80	1.33	1.76	2.09	2.45	2.52	2.86	2.86

Table 2

The data in Table 2 shows the enrollment, in millions, in collegiate mathematics offerings (excluding statistics and computer science courses) since 1960. The concave down pattern in the data again suggests a power function model. A power function that fits this data well is $N(t) = 0.1300\ t^{0.8415}$, with a correlation coefficient of $r = 0.9824$. This power function for total mathematics enrollment, with power $p = 0.8415$, is growing slightly more rapidly than the power function for total enrollment where $p = 0.8053$.

For comparison, the corresponding exponential function that best fits this data is $M(t) = 0.775(1.034)^t$, where $t = 0$ corresponds to 1950. The growth factor 1.034 indicates an annual growth rate of about 3.4%, so that, under the two exponential models, the growth in mathematics enrollment has proceeded less rapidly than overall college enrollment. The corresponding correlation coefficient is $r = 0.9228$. Both the power function (solid) and the exponential function (dashed) are shown in Figure 2.

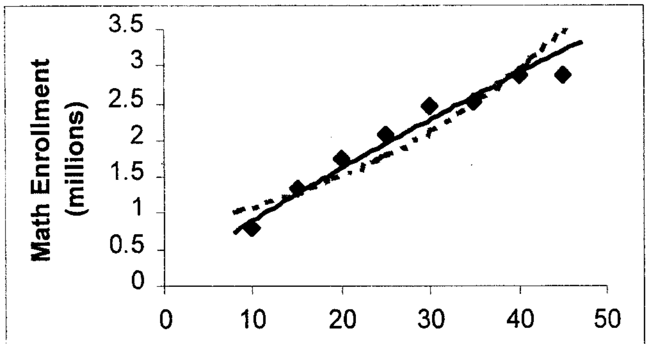

Figure 2

It is interesting to note that the conclusion drawn about which process has grown faster can depend on the model chosen.

Actually, the flattening in the data seen in the 1980-85 period (from $t = 30$ to $t = 35$) and again from 1990 to 1995 (from $t = 40$ to $t = 45$) suggests that a better fit might be achieved using a quartic polynomial. From a more sophisticated point of view, we should expect that this would be a much better fit because quartics are a five-parameter family of functions, while exponential and power functions are two-parameter families. The additional three parameters provide three extra degrees of freedom and so should lead to a significantly better fit to the data.

Year	1950	1960	1970	1980	1990	1999
Population	150.7	179.3	203.3	226.5	248.7	272.0 est

Table 3

Next, Table 3 shows the total U.S. population, in millions, since 1950. The best-fitting exponential function is $P(t) = 87.03(1.0118)^t$. This indicates that the U.S. population has been growing at an annual rate of 1.18%. This is considerably slower than the rate of growth of either the entire college-level enrollment or the mathematics enrollment. The correlation coefficient $r = 0.9929$ indicates a very high level of positive correlation and, from the graph in Figure 3, we see that the fit is quite good.

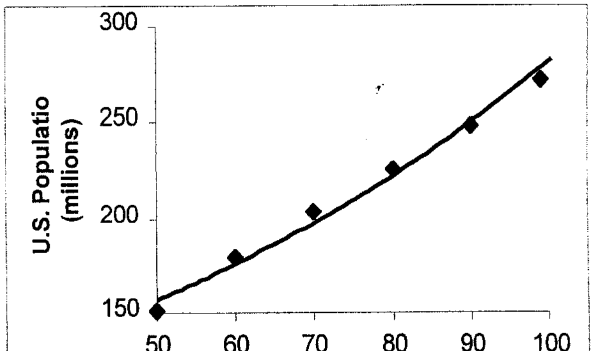

Figure 3

There are several significant implications of the above information, perhaps the most important being from a demographic point of view. Fifty, and even 30 years ago, the students coming to college were a very small portion of the total U.S. population; from a traditional mathematical perspective, they were an elite group who had mastered a high level of proficiency in traditional high school mathematics, particularly algebraic manipulation. They entered college reasonably well prepared for the standard freshman course in calculus, which tended to have an algebraic focus for the subject. More recently, as the cadre of college bound students has increased dramatically, they can no longer be viewed as an elite group. Certainly, a reasonable percentage of today's students are comparable to the elite of the past, but those students tend to attend the elite postsecondary institutions. As a consequence, the student population that the majority of U.S colleges face consists predominantly of students who increasingly have not mastered traditional high school mathematics; in turn, ever greater proportions of them are being placed in some kind of remedial track designed to develop all the traditional algebraic skills that once were necessary for a traditional calculus course.

But, freshman calculus courses in the U.S. have been undergoing significant change in the last decade as the focus of a nation-wide reform movement. These reform calculus courses seek to achieve:

- a balance among graphical, numerical, and algebraic approaches to the subject,
- an emphasis on conceptual understanding rather than rote manipulation, and
- a focus on realistic applications from the point of view of mathematical modeling, often through an early introduction to differential equations.

Much of this can be achieved only through the availability of sophisticated technology, most commonly graphing calculators, although some schools make heavy use of computer software such as Derive, Maple or Mathematica. An analysis of the status of this movement is in [1]. A discussion of the challenges to be met in the forthcoming decade is in [2].

But, if the students in college-level math courses are using sophisticated technology to assist in learning and applying the mathematics, certainly the practitioners who actually use mathematics today (let alone in the coming decades) are utilizing technology that is at least as sophisticated. In reality, any routine operation that people use repeatedly has already been programmed. It therefore makes little sense to focus math courses primarily on making students into imperfect organic clones of a $150 graphing calculator with CAS (Computer Algebra System) capabilities! The students will never win the competition — they will never be as fast or as accurate as the machine. The question we all need to consider is: What should be the focus of mathematics education? The authors believe that the mathematical discussion at the beginning of this article is the answer. Very few people today, let alone in the future, will need to factor anything as complicated as $x^8 - y^8$. However, virtually any educated individual will need the ability to examine a set of data and recognize a behavioral pattern in it, to assess how well a given functional model matches the data, to recognize the limitations in the model, to use the model to draw appropriate conclusions, and to answer appropriate questions about the phenomenon being studied. In turn, this requires

- A deep understanding of the function concept, function notation, and the meaning of variable,
- A knowledge of different families of functions, including being able to distinguish between different families of functions graphically, numerically and algebraically,
- A knowledge of the behavior of the different families of functions depending on one or more parameters,
- The ability to select the appropriate tool to solve the equations that arise from using the models, be it pencil-and-paper, graphing calculators, spreadsheets, or CAS systems.

- The ability to interpret mathematical ideas and to communicate these ideas to others.

More extensive discussions of reform efforts in the courses that precede calculus are in [3].

The *Functioning in the Real World* Project In this article, the authors will describe a project that developed and implemented an innovative course based on the preceding ideas and will discuss a variety of assessment questions related to it. The course, called *Functioning in the Real World* [4], which is based on the idea of mathematical modeling, was developed with support from the National Science Foundation, and is in the spirit of the calculus reform movement. The course focuses on mathematical concepts, provides students with an appreciation of the importance of mathematics in a scientifically oriented society, provides students with the skills and knowledge they need for subsequent mathematics and other quantitative courses, makes appropriate use of technology, and fosters the development of communication skills. The goal is to emphasize the qualitative, geometric, and computational aspects of mathematics within a framework of mathematical modeling at a level appropriate to precalculus students. The *Functioning* course emphasizes mathematical topics such as families of functions, data analysis and fitting functions to data, difference equation models, modeling periodic phenomena, matrix algebra and its applications, and probability models. In the process, students are encouraged to view mathematics and its applications from graphical, numerical, and symbolic perspectives with a focus on conceptual understanding. Student project assignments, writing, use of technology, and collaborative learning are all emphasized throughout. All of these are major themes that permeate both the NCTM (National Council of Teachers of Mathematics) Standards [5] and the AMATYC (American Mathematical Association of Two Year Colleges) Crossroads Standards for Introductory College Mathematics Courses [6], the two principal documents distributed nationally in the U.S. with guidelines for teaching modern courses at the high school and introductory college level. In fact, the *Functioning* course was named as an exemplary project for implementing the AMATYC Standards [7]. It was also named the best project in the algebra through precalculus category in the INPUT (Innovative Programs Using Technology) program [8] sponsored by the Annenberg Foundation/Corporation for Public Broadcasting.

The *Functioning* course capitalizes on the fact that most students are more interested in the applications of mathematics than in the mathematics itself, so that the applications drive the mathematical development. All the mathematical knowledge and skills that students will need for calculus, especially reform calculus, are introduced and reinforced in the process of applying mathematics to model realistic situations and to solve interesting problems that arise naturally from the contexts. The authors and others who have used the materials have found that this approach excites the students and encourages them to go further with mathematics by showing them some of the value that mathematics provides in today's world.

The *Functioning* materials serve a variety of audiences. First, they can be used for a one or two semester course that lays a different, but very effective, foundation for calculus. Second, they provide a one semester contemporary capstone to the mathematics education of students who do not plan to continue on to calculus. (At most U.S. colleges, a course in traditional manipulative algebra typically plays that capstone role.) The authors have seen that the *Functioning* course actually encourages many of these students to change their minds and to go on to calculus and other quantitatively related courses. Some students have even changed their majors to become math majors as a result of the experience.

The fundamental idea around which the *Functioning* course is based is the function concept in all of its manifestations with emphasis on its applications to the world around us. Curve fitting techniques are introduced early and used throughout so that students see how to apply functions in many different contexts, just as the authors did in the Background discussion at the beginning of this article. The data analysis methods simultaneously provide immediate reinforcement regarding the behavior and properties of the different families of functions studied. The focus on difference equations includes applications of first and second order difference equations and systems of first order difference equations that sets the stage for differential equations in later courses. The emphasis is on modeling a variety of situations, interpreting the behavior of the solutions in terms of the situations, and looking at the effects of changes in parameters. Matrix algebra is introduced as a unifying tool for investigating a wide array of applications such as systems of difference equations, Markov processes, and geometry.

The trigonometric functions are introduced from the point of view of modeling periodic phenomena. For example, students are asked to construct a sinusoidal function to model the temperature in a house where the furnace comes on when the temperature drops to 66° and turns off when the temperature reaches 70°, a cycle that repeats every 20 minutes. As another example, the students are asked to model a person's blood pressure over time given readings of 120 over 80 and a pulse rate of 70.

The following is an annotated description of the contents of the project materials.

Functions in the Real World introduces students to the function concept from graphical, numerical, symbolic, and verbal points of view as functions arise in daily life. The emphasis is on the behavior of functions (increasing or decreasing, concave up or concave down, point of inflection, periodicity).

Families of Functions includes linear, exponential, power, and logarithmic functions, with emphasis on their applications and their qualitative behavior, in the spirit of the calculus reform movement. The intent is to have the students learn how to identify and distinguish the different families from algebraic, graphical, and numerical representations.

Fitting Functions to Data includes linear and nonlinear curve fitting to reinforce the properties of the different families of functions, to develop algebraic skills in working with the properties of those functions, and to connect the mathematics to the real world.

Extended Families of Functions includes polynomial functions, fitting polynomials to data, the nature and relative frequency of the roots of polynomial equations, building new functions from old (shifting, stretching, sums, differences, products, quotients, and composition of functions) , finding roots of equations, and finding polynomial patterns (including sums of integers and sums of squares of integers).

Modeling with Difference Equations includes the development of models for describing population growth, logistic (inhibited) growth, eliminating drugs from the body, radioactive decay, Newton's laws of heating and cooling, geometric sequences and their sums, financing and amortization, fitting logistic curves to data, iteration and chaos, etc.

Modeling Periodic Behavior stresses using trigonometric functions to model phenomena such as the number of hours of daylight as a function of the day of the year, temperatures over the course of a year, and tides; relationships between trig functions; approximating periodic functions with sine and cosine terms; approximating the sine and cosine functions with polynomials; properties of complex numbers; and chaotic phenomena.

Matrix Algebra and its Applications includes a variety of applications of matrices, such as Markov chains in the spirit of a finite math course, not merely the use of matrices for solving systems of linear equations.

Probability Models includes binomial probability and the binomial expansion, geometric probability, estimating areas of plane regions using Monte Carlo simulations, waiting time models, the spread of epidemics, and random patterns in chaos.

Systems of Difference Equation includes the predator-prey model, an arms-race model, a labor-management model, a model for marriage rules among the Natchez Indians, and matrix growth models leading to a treatment of the eigenvalue/eigenvector problem.

Geometric Models includes analytic geometry, the conic sections, parametric curves, the average value of a function, and curves in polar coordinates.

Changes in Pedagogy The *Functioning* course almost forces a change in pedagogy; it is virtually impossible to give the course in a formal lecture format. The non-routine nature of many of the problems make them ideal for having the students work together in small groups using collaborative learning.

The authors also emphasize the use of individual or small group projects related to the mathematical content of the course. For example, students are required to find a set of data of interest to them and perform a complete analysis of it -- finding the best linear, exponential, and power function to fit the data, and asking and answering pertinent questions (i.e., predictions) based on the context. Each student is required to write a formal project report. For instance, during a recent semester, a sample of the topics studied by the students in one of the author's classes include:

The number of sexual harassment cases filed as a function of time.
The likelihood of car crashes as a function of blood alcohol level.
The growth of the prison population as a function of time.
The time of high tide at a beach as a function of the day of the month.
The amount of solid waste generated per person as a function of time.
The time for water to come to a boil as a function of the volume of water.
The size of the human cranium over time during the last three million years.
The results of a serial dilution experiment in biology lab.
The growth in the Dow-Jones average as a function of time.
The Gini Index measuring the spread of rich versus poor in the population over time.
The number of immigrants who entered the U.S. over time.
The mean annual income as a function of the level of education.

In the process of writing such reports, the students must decide on which variable is independent and which is dependent. They must come to grips with the practical meaning of domain and range as the limitations inherent in the model they are creating; it is not just a matter of avoiding division by zero, but rather a matter of a high level of mathematical judgment as they connect the mathematics to the real world. They must understand the practical meaning of the slope of a line, not just think of it as a ratio of the number of boxes in two directions.

For another project, students are given a set of data of a periodic nature, say historical high temperature readings in a given city every two weeks, and are asked to create a sinusoidal function that models the temperature. Initially, this can be quite a challenging problem, and yet in the process the students really come to understand the meaning of the parameters in the general equation for a sinusoidal function. A particularly effective way to start this project is to have the students work in small groups, so that someone has to speak up and make some initial suggestion of how to begin. Eventually, each student must complete the analysis independently and prepare a formal written report.

The fascinating thing about such a project is that there are several different strategies that can be developed for estimating the various parameters and different students come up with them. One of the most memorable lines by a recent student in one such report was: "*The next quantity to be determined is the frequency. This was deceptively simple.*" How often does a student in a precalculus course describe the frequency of a sine function as "*deceptively simple*", particularly when the value he obtains for the frequency is .0172 or $2\pi/365$?

The Role of Technology The authors believe that technology has tremendous implications for the teaching and learning of mathematics. The problem mathematics educators face is how to use the available technology in the service of the mathematics rather than as an end in itself. To illustrate this dichotomy, consider the question of fitting functions to data. Most graphing calculators have the ability to find the best linear, exponential, power or logarithmic function to fit a set of data; they also fit the best quadratic, cubic or quartic polynomial; some can fit the best sinusoidal or logistic function to a set of data. All of this can be accomplished literally at the push of a button.

However, the authors firmly believe that pushing that button, at least early on, is a mistake for most precalculus or college algebra students. Rather, students should learn to examine the original set of data, look for a general pattern, appropriately transform the data entries to linearize them, use a calculator or computer to obtain the best linear fit to the transformed data, and then undo the transformation using the algebraic properties of the appropriate inverse function. In this way, the students are learning more mathematics while developing some of the manipulative skills they will need to succeed in calculus and other courses.

Assessment A wide variety of assessment activities connected with the *Functioning* project have been conducted over the last several years, most as part of the project's original NSF funding. Recently, as part of a separate NSF-supported project to encourage interconnections between mathematics and all other disciplines using mathematics, that project's external evaluator conducted a series of attitudinal surveys comparing student responses in reform courses to those in courses that were not being affected by the grant project. As one component of this evaluation effort, a study was conducted at Suffolk Community College (a large comprehensive two-year college). The following results comparing students' responses to a series of statements in the *Functioning* course to students' responses in traditional college algebra/precalculus sections were obtained:

- "*This course raised my enthusiasm for math*", 76% of the students in the *Functioning* course agreed while 38% of the students in the traditional course agreed.
- "*Because of this course, I plan to take more quantitative courses in the future.*", 64% of the students in the *Functioning* course agreed while 42% of the students in the traditional course agreed.
- "*This course required me to ask: Why?*", 88% of the students in the *Functioning* course agreed while 43% of the students in the traditional course agreed.
- "*This course helped me understand how to apply math to real world problems.*", 84% of the students in the *Functioning* course agreed while 62% of the students in the traditional course agreed.
- "*I learned how to use concepts to solve problems.*", 92% of the students in the *Functioning* course agreed while 52% of the students in the traditional course agreed.
- "*This course improved my problem solving skills.*", 80% of the students in the *Functioning* course agreed while 62% of the students in the traditional course agreed.
- "*Because of this course, I plan to take more quantitative courses in the future.*", 64% of the students in the *Functioning* course agreed while 42% of the students in the traditional course agreed.

A comparable attitudinal survey was conducted at New York Institute of Technology (a moderate-sized private career-oriented college) this past semester. In this study, four different instructors taught a precalculus course. Two of them (A and B) taught a very traditional, highly algebraically oriented course using graphing calculators; the other two (C and D) taught a reform course using the *Functioning* text. The external evaluator used a set of 20 attitudinal questions that were administered on the first day of class (pre-survey) and again on the last day of the semester (post-survey) to investigate the extent to which student attitudes changed as a result of each course. The questions were clustered in several general areas. For the analysis, he combined the Strongly Agree and Agree responses, as well as the Strongly Disagree and Disagree responses.

	Pre	Post	Change
Traditional A	57.3	47.2	-10.1
Traditional B	56.0	41.7	-14.3
Reform C	58.0	66.7	+8.7
Reform D	60.5	63.2	+2.7

Table 4

The first cluster of questions dealt with whether mathematics is an active, open-ended, discovery-oriented process or a passive, closed-ended, memory-based procedure. The results are shown as percentages of positive responses in Table 4. The pre-course responses are clustered close to the mean, with a range of only 4.5 and a standard deviation of 1.9, so that the four groups are comparable at the outset. The post-course responses, however, are not clustered, with a range of 25 and a standard deviation of 12.1. Group D showed a modest increase; group C showed a substantial increase in the direction of more positive attitudes and experiences; Groups A and B showed substantial movement in a negative direction regarding attitudes and experiences about mathematics as an open, active, discovery-based process.

The second cluster of questions dealt with the utility of mathematics and if the students view it as being connected to situations beyond math courses. Some of the survey items were:

This course helped me to understand how to apply math to real world problems.

In this course, I learned ways of thinking that are useful in situations outside of math.

This course showed that math is useful in many non-math courses.

This course made connections across disciplines.

The corresponding percentage responses for the four groups are shown in Table 5. Percent of positive responses for the pre-course survey are again clustered, although not as closely as in the previous area, with a range 13.5 and a standard deviation of 6.1. Post-course results are scattered even more widely than in the first area, with a range of 62.3 and a standard deviation of 28.3. Once again, groups C and D diverge from groups A and B in pre/post scores. Average pre/post means for the two reform text classes moved from 55.4 to 53.2, while average pre/post means for the traditional text classes fell from 58.75 to 9.25. Based on the responses to the two general areas summarized above, the evaluator concluded that students appear to respond much more positively to the reform approach than to the traditional approach. These findings are useful because the reform calculus and precalculus texts and methods emphasize these areas of active learning and applied/connected math. The evaluator hypothesized that differences in the student populations or in the instructors might account for the differences. Based on responses to other items, however, the evaluator found that he can, to some extent, reject these hypotheses.

	Pre	Post	Change
Traditional A	65.5	6.0	-59.5
Traditional B	52.0	12.5	-39.5
Reform C	57.5	68.3	+10.8
Reform D	53.3	38.0	-15.3

Table 5

A third cluster dealt with the importance of technology in learning mathematics. The results are shown in Table 6. The evaluator noted that the average for group C was the lowest in the pre-course survey but increased to 100 percent positive rating, while the other three groups all declined in percent of positive response.

	Pre	Post	Change
Traditional A	78	67	-11
Traditional B	87	70	-14
Reform C	76	100	+24
Reform D	88	79	-9

Table 6

It should be noted that, in all these areas, there were differences in the student responses in the reform group C compared to the reform group D. This may be accounted for by the fact that the instructor for group C was totally committed to the reform approach while the instructor for group D found it hard to relinquish some of the traditional manipulative topics and attempted to incorporate many of them into the course, so the course actually became a mixture of reform and traditional topics.

A further component of the study at NYIT involved comparison of student performance in the succeeding calculus course, a reform offering based on the Harvard Calculus project text, which is used in all sections of the calculus courses. On the first class test in calculus, the students coming from the traditional approach had a mean grade of 64.36 with a standard deviation of 19.23. In comparison, the students coming from the reform sections had a mean of 90.62 with a standard deviation of 8.96. When the difference of means of the two groups is analyzed statistically, the associated t-value is $t = 4.71$ standard deviations and the associated p- value is $p = 0.0000257$, so the differences between the groups are statistically significant.

It is interesting to note that the test scores in the group with the traditional precalculus background ranged from a minimum of 32 to a maximum of 97. The scores in the reform group ranged from a minimum of 75 to a maximum of 103. Note that the mean of the scores for the traditional group (64.36) is well below the minimum score of the students in the reform group (75).

The results on the second and third class tests in calculus demonstrate similar, though not as dramatic, patterns. On the second test, the students coming from a traditional approach scored a mean grade of 59.58 with a standard deviation of 21.82. The students coming from the reform sections scored a mean grade of 77.42 with a standard deviation of 14.26. When the difference in the means of the two groups are analyzed statistically, the associated t-value is $t = 2.61$ standard deviations and the associated p- value is $p = 0.01249$. The difference is statistically significant at all levels. On the third class test, the students with a traditional background scored a mean grade of 62.3 with a standard deviation of 17.31, while the students with the reform background scored a mean grade of 76.1 with a standard deviation of 18.13. The t-value for this difference of means is $t = 2.20$ and the associated p-value is $p = 0.035$. The difference is statistically significant at the 5% and 10% levels of significance.

Furthermore, the average scores (out of 20 points each) for the students in the two groups on a set of short quizzes administered throughout the semester also indicate a consistent pattern, as shown in Figure 4. The group with the reform background consistently averaged higher scores on each quiz. Incidentally, the quizzes on which the students with the traditional background came closest to matching the students from the reform approach are the quizzes that were primarily algebraic in content. Despite that, the students from the reform background still outperformed the ones from the traditional background.

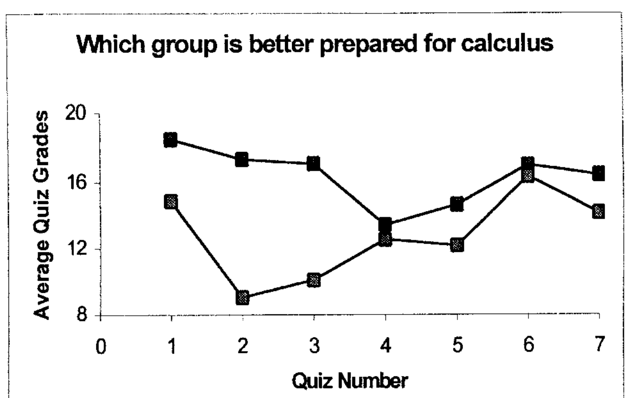

Figure 4

On the final exam in Calculus, the students coming from a traditional approach scored a mean grade of

55.8 with a standard deviation of 23.09. In comparison, the students coming from the reform sections scored a mean grade of 69.8 with a standard deviation of 26.21. When the difference in the means of the two groups is analyzed statistically, the associated t-value is t = 1.64 standard deviations and the associated p-value is p = 0.11 The difference is statistically significant at the 10% level of significance.

It is interesting to note that 12 of the 13, or 92.3%, of the students from the reform background who started the course were still around for the final exam. In comparison, 24 of the 39, or 61.5%, of the students from a traditional background who started the course were still around for the final.

Actually, the comparison of student performance on the last few quizzes, the last class test and the final exam is somewhat distorted considering the disproportionate number of students with a traditional background who stopped attending the calculus course. Since they were already doing poorly in calculus, their scores on the last class test and the final, had they persisted, would undoubtedly have lowered the average considerably for their group. That is, the lowest performing students were effectively removed from the traditional group when they stopped attending, but the lowest performing students from the reform group completed the entire course. The only student in the reform group who withdrew from the course was scoring reasonably high grades, but had to drop the course for purely personal reasons. Had all these students persisted to the end of the semester, there would almost certainly have been an even wider discrepancy in the averages between the two groups.

The issue of student persistence and retention in calculus is another critical factor. Only one student from the reform background stopped attending the course, no other student from this grouped missed a single test and, in fact, only 2 of the others missed any quizzes. On the other hand, over 40% of the students from the traditional background stopped attending, 6 of them skipped the first test, 8 of them skipped the second test, 16 missed the third test, and more than three-quarters of them have missed one or more quizzes.

Finally, 10 of the 13 students from the reform background who started the course received passing grades in calculus, for a success rate of 76.9%. In comparison, only 16 of the 39 students with a traditional background who started the course received passing grades in calculus, for a success rate of 41.0%.

These evaluation results at NYIT are a portion of a much larger assessment project, but the balance of the study has not yet been completed by an external evaluator.

The second author also had the opportunity, several years ago, to teach two parallel sections of the "same" precalculus course, one from the *Functioning* materials and the other from a more traditional text, to appease a conservative department chair. She decided to ask some common questions on the two final exams to compare student performance. However, she felt it would be unfair to the students in the traditional class to ask any conceptual or realistic applied problems. Therefore, as the common questions for both groups, she only posed routine (mechanical) questions or posed the routine part of a realistic application to the traditional group. For example, the following question appeared on the exam for the *Functioning* class:

> At a certain pier, the low water line is 6 feet above sea bottom and the high water line is 14 feet above bottom. If low tide occurs at midnight and high tide at 6 am,
>
> (a) What are the amplitude, frequency and period for this function?
>
> (b) Sketch the graph, including appropriate scales.
>
> (c) Write an equation of the water height H as a function of time t.
>
> (d) How high is the tide at 11 am?
>
> (e) When is the water 12 feet deep?

The students in the traditional course were asked:

> Let $y = f(t) = 10 + 4 \cdot \sin[(\pi/6)(t - 3)]$.
>
> (a') Find the amplitude, frequency and period.
>
> (d') Find $f(11)$.
>
> (e') $10 + 4 \cdot \sin[(\pi/6)(t - 3)] = 12$. Solve for t.

For purposes of comparison, the same number of points were allotted to parts (a) and (a'), (d) and (d'), and (e) and (e'). Out of 11 points, the students in the traditional course scored an average of 3.9 while the students in the *Functioning* course scored an average of 9.6. Based on a small sample t-test for the difference of means, this represents a value of $t = 7.014$.

Notice that the students in the *Functioning* course had to understand the context, translate it to a mathematical model, and create the formula that was simply handed to the students in the traditional course. Further, they had to interpret the height of the tide at 11 am as representing $H(11)$. Finally, they had to interpret the meaning of "when is the water 12 feet deep?" as requiring them to set up and solve the equation $10 + 4 \cdot \sin[\pi(t - 3)/6] = 12$, which was simply handed to the other group.

Incidentally, the students in the *Functioning* course scored better on six of the seven common questions on the two exams, four of them being statistically significant. A full analysis of the results of this study appears in [9]. A similar study comparing student performance on common questions on the precalculus final exams for Fall 1999 is still in the process of being conducted by an external evaluator.

Working in a realistic context as opposed to an abstract setting appears to make all the difference. The mathematical ideas make sense and the greater mathematical expectations placed on the students becomes acceptable to them. On the other hand, when students in traditional classes are assigned 50 indistinguishable problems every night, all of which look like the same things they think they've seen before, the students are not likely to do many homework problems at all.

The assessment results reported above based on the authors' personal experiences mirror comparable experiences reported by other faculty who taught the *Functioning* course. During the development phase of the project, under National Science Foundation funding, instructors from a wide variety of institutions – high schools, two year colleges, private and public four year colleges, technical colleges, and large research universities – taught from preliminary versions of the materials and their suggestions and experiences were incorporated into the final version. In some ways, assessment information obtained from such individuals who class-tested the project materials can be more significant, since they were not directly involved in the development of the materials. Although most of the information received from the class-testers was anecdotal in nature, it is often highly indicative of what went on in their classes. For instance, there were reports of:

- Higher levels of student interest in the course.
- Higher levels of student attendance.
- Higher success rates and higher persistence rates, especially among students who have never previously had great success in mathematics.
- High levels of success in subsequent courses.
- High levels of approval from faculty in other disciplines who appreciated the applications-driven approach.
- Students who changed their minds and took more mathematics courses because of their experience in the *Functioning* course.
- Students who changed their majors to become math majors because of the course.

Perhaps most telling, however, are actual written comments made by students on formal course evaluations. For instance,

> "*Math is a part of life -- everything we experience, whether tides or hours of daylight or population growth or rising medical costs, deals with math. I now have a much better understanding of the patterns of life and how math can be applied to them.*"
>
> "*My overall reaction to the course was extremely positive. By emphasizing the value of mathematical pursuits through applications first (and theory being derived from the application), the course proved to be constantly interesting. Past math courses seemed tedious. This course never struck me as tedious (challenging? very, but not tedious). Nice to think of math as an intellectual pursuit, a very useful tool, and a way of seeing life -- as opposed to 'a course I have to take to complete a chemistry curriculum'. Above all, being able to see how I could use the subject I was learning in the 'real' world.*"

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