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CHAPTER 6

Renewing the Precursor Courses

New Challenges, Opportunities, and Connections

Sheldon P. Gordon

1. INTRODUCTION

The earth's crust is composed of a series of large plates floating on the underlying molten magma. These plates are constantly shifting and consequently they bump into one another and often one will ride up on top of another near their edges. The interfaces between the plates form fault lines in the earth and the resulting pressures that build up along the interfaces eventually release to form earthquakes.

The mathematics curriculum can be viewed in much the same way as being composed of a series of such plates. Historically, the two largest—and perhaps the most important—are the secondary and college mathematics curricula. Over the last several decades, the college-level plate has split into two, one of which includes the traditional first two years from calculus through differential equations and linear algebra and the other includes the developmental courses designed to replicate the secondary-level curriculum for either students who avoided mathematics in high school or students who need to learn the material again. However, e are associated plates in other disciplines, such as physics, engineering, ogy, and economics, as well as one for upper-division and graduate level hematics. All of these plates affect the plate for the first two years of college hematics.

For most of our careers, the primary mathematical plates—developmental, first two years, and the upper division—have been quite stable. The under- g magma on which they ride has been solidified, so there has been relatively movement. In particular, the interface between the first two plates has been e smooth. The high school curriculum was well established and mathemati- ıs were very cognizant of its content, having experienced it firsthand. ısequently, mathematicians could count on its invariance in the development urricula at the college level. When we designed courses that are precursors to ulus—developmental algebra, college algebra, and precalculus—they were entially clones of the high school equivalents, though sometimes with the viso to do it "faster and louder."

A major part of the comfort level with this curriculum is that mathematicians w what it does well—by replicating the experiences they went through, this iculum produces clones of themselves. Unfortunately, only a very small centage of the students in these courses today have the capability, or the desire, it that mold. And worse still, the value placed on people with this kind of ning is diminishing. At the same time, these courses do not provide the hematical needs of the other students; there is a good reason why, nationally, ollment in mathematics drops by about 50% in each successive course from h-grade algebra on up.

Now, the entire structure of mathematics education is in the process of nge. The postsecondary curriculum is undergoing a series of dramatic nges. The calculus reform movement has made some significant and far- hing changes. As a result, consequent changes are needed in the subsequent rses, such as multivariate calculus and differential equations, as well as in the cursor courses that prepare students for calculus. There are growing pressures ncrease students' exposure to statistical ideas and probabilistic reasoning early he curriculum; likewise, there are pressures to introduce matrices sooner. at then of the interface between the secondary curriculum and the college riculum or that between the precursor curricula that exist at the college level the mainstream curriculum?

CHANGES IN THE UNDERGRADUATE CURRICULUM

Think about the overall structure of the courses that prepare students for ulus in the traditional mathematics curriculum. In Algebra I we teach most of fundamental rules and methods of manipulative algebra. While *we* teach it,

they don't learn it. Therefore, at least 80% of Algebra II is devoted to repeating the same topics. So, why does the follow-up course in college algebra, or precalculus for that matter, still reteach the same rules and methods of algebra? Because, although we still teach it, most of the students still haven't learned it. Of course, there are always a few exceptions—those who are exceptionally good at the manipulations. The mathematicians teaching these courses are among those exceptions.

There was a lot of truth in the old adage "you take calculus to learn algebra." In traditional calculus, students were routinely expected to *use* algebra—often for the first time—to solve substantial problems or for lengthy derivations. It was no longer just a meaningless game of drill for no apparent purpose; if students couldn't do the algebra, then they couldn't do calculus. Unfortunately, the emphasis in too many calculus courses shifted to endless drill and practice on long lists of derivatives and integrals. Thus, in the process of finally learning algebra, most students came away with the perception that calculus primarily consists of facts such as the derivative of x^3 is $3x^2$. Relatively few students ever understood what the derivative of a function really meant, even though they could differentiate quickly. So there was the corollary to the above adage, "you take differential equations to learn calculus."

The calculus reform movement insists that students should learn and understand the fundamental concepts of calculus while taking calculus. The reform projects place far greater emphasis on conceptual understanding of the ideas of calculus, achieve a better balance among graphical, numerical, symbolic, and verbal representations, and incorporate more realistic applications, often from the point of view of mathematical modeling with differential equations.

The changes in calculus, while originally envisioned as being primarily content changes, quickly included pedagogical changes as well, with frequent emphasis on more active learning environments (such as collaborative and cooperative learning), use of individual and group projects, and emphasis on writing and communication. In order to accomplish this, however, there has been a concomitant reduction in the traditional emphasis on symbolic manipulation. Most of the instructors teaching these courses believe that the courses are the better for these changes since they focus on calculus and its value; they believe that the students are the better for it because they are learning calculus and how to apply it; and they feel that they personally are the better for it, since they are finally teaching mathematics and not simply more algebra. (Of course, there are certainly some who believe that a de-emphasis on algebraic manipulations is a dreadful loss.)

The changes in both the content and pedagogy in calculus have set the stage for comparable changes in differential equations and other postcalculus courses. Simultaneously, the new calculus courses are also having an impact at the secondary level as the Advanced Placement (AP) calculus course has changed the same ideas and goals. Finally, comparable changes are working their those college courses that were the traditional calculus preparatory track w[1]). A variety of reform projects have resulted in alternative courses at lculus, college algebra, and developmental algebra levels that likewise s emphasis on many of the routine algebraic skills, particularly those d with factoring polynomials and operations with rational expressions. these new courses focus more on conceptual understanding of the ntal mathematical ideas—variable, function, behavior of functions—as on realistic applications of the mathematics. They often feature "new" tical content, such as the use of real-world data (once thought to be the of statistics) and the notion of fitting a function to the data, or aspects of ity, or recursion and iteration, or substantial applications of matrix that go well beyond solving systems of linear equations.

many ways, this reflects the new paradigm for the mathematics that is used in practice. In several presentations, including one at the U.S. Academy (West Point) conference on the Core Curriculum (see), Henry Pollak has used an illustration such as that in Fig. 1 to dramatize t away from the mathematics used by practitioners (engineers, scientists, mathematicians, and others who used the mathematics) in 1960, say, ost of today's senior mathematics faculty were in school. At that time, every problem considered by the practitioners was continuous and was hed from the point of view of seeking a closed-form, deterministic . Relatively few problems were discrete in nature; some were stochastic ense of having a random component. The mathematics curriculum of the ically mirrored this paradigm closely.

day, a very different paradigm exists in terms of the mathematics used, as in Fig. 1—virtually every real-world problem now has a major discrete ent, if it is not inherently discrete; even continuous models must be zed to permit computational solutions. Most realistic problems today have om component—there is always some degree of uncertainty. Yet, the atics curriculum today does not reflect these new emphases. For the most e content is the same as in 1960, although there is slightly more emphasis

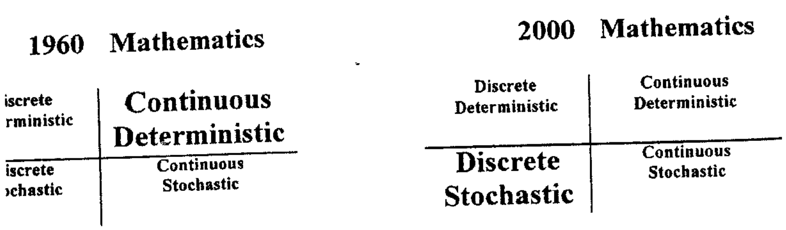

Figure 1. The changing face of the mathematics used in practice.

on statistical reasoning and some discrete topics now exist in the curriculum. One of the major challenges mathematicians face is integrating more of these ideas and methods into the curriculum at all levels (see Dossey[2]). In the courses that are precursors to calculus, the larger challenge is to do this while preparing students for calculus.

At the same time, these precursor courses are increasingly unsuccessful for a variety of reasons that will be discussed later. An analysis of data in a report by the Conference Board on the Mathematical Sciences[3] indicates that perhaps 10% of the students who take college algebra or precalculus courses in college ever go on to calculus. Many students take these courses to fulfill a quantitative literacy requirement. Many students planning to continue do so poorly in these courses that they "leak out" of the mathematics pipeline. Perhaps it makes sense to rethink the intent of such mathematics courses and to make them valuable in their own right, rather than to think of them as courses designed exclusively to prepare students for calculus.

3. CHANGES IN THE SECONDARY CURRICULUM

While most college-level mathematics faculty have naturally focused on the changes that are being felt at the college level, very significant forces have been transforming the secondary curriculum as well. If anything, this transformation has been happening for considerably longer than those at the college level. Most college faculty have been aware of them, but only peripherally. We decry the fact that incoming freshmen appear to have poorer manipulative skills and less of the information that has traditionally been considered important for success in college level mathematics. Based on what can be inferred from dealing with these students and based on our own high school experiences, most college mathematics faculty typically conclude that either the students are academically worse or that the high schools are completely at fault. In turn, the colleges have reacted by introducing and expanding their developmental and other precursor offerings to bring the students up to speed. At many institutions, these precursor programs are the majority of the mathematics offered, with more sections, more students, and more institutional resources expended than in any other area.

Perhaps, though, the problem really results from a shift in the secondary curriculum plate, implying that the smooth interface we have always expected is no longer there. All mathematics faculty have heard about the *Curriculum Standards* from the National Council of Teachers of Mathematics (NCTM),[4] but few have paid great attention to them and far fewer have ever read them. Yet, the *Standards* is having an ever-increasing impact on *what* is taught in the secondary schools and *how* it is taught.

The *Standards* call for a very different approach to teaching mathematics, ıhasizing a greater depth of understanding of mathematical concepts and oning. They also emphasize geometrical and numerical ideas as a balance to ɛly symbolic ideas, as well as the use of substantial applications of the hematics, often in the context of group projects. The *Standards* require eased communication on the part of students in the form of written and oral ɔrts. They often involve collaborative learning activities. They also presume : technology assumes an appropriate and ongoing role in the teaching and ning of mathematics and, in reality, the use of graphing calculators is quickly oming universal since they were mandated for the AP calculus exam ;inning in 1995 and are allowed on almost all other standardized tests. The *ndards* calls for the early introduction of many different mathematical ideas ɔ the curriculum, particularly statistical reasoning and data analysis, matrix ebra and its applications, recursion and iteration, and probability. Overall, plementation of the *Standards* implies higher expectations for teachers and dents. And, in actuality, these goals and expectations match closely the goals d expectations of the reform efforts at the college level.

However, something has to be removed to make room for all these new ıphases. In the process, therefore, the *Standards* call for a diminished emphasis traditional formal algebraic manipulation. For example, students spend nsiderably less time factoring polynomials. Instead, students are expected to ıderstand the notion of the roots of an equation, which they apply to factor nple expressions. They learn how to determine the real roots of more ımplicated equations graphically and numerically and then use these roots as ɛeded. Presumably, these students will have far more understanding of the ɔncept of a root; however, the cost is that they may have somewhat less facility recognizing factors at a glance.

Is this a fair trade-off? *In principle*, most faculty at the college level will elcome students with such backgrounds. Most of these changes are completely ɔmpatible with the spirit of the calculus reform movement and with the changes ıat are beginning to be implemented in courses before and after calculus. *In ractice*, however, things are somewhat different. All of the mechanisms that urrently bridge the relatively small gap between the secondary school and the ıtroductory collegiate plates now have to be rethought.

These comments also assume that such new courses exist widely in the econdary schools. How extensive are they in actuality? From discussions with ndividuals who have developed materials for such courses, it is estimated that at east one-third of all new textbooks purchased by the secondary schools reflect ıew curricula. The figure may be much higher, particularly as new editions of raditional texts incorporate more of the reform ideas. In addition, many teachers ıre implementing some of the ideas and themes in the absence (mostly for ɔudgetary reasons) of formal sets of textbooks.

4. COLLEGE PLACEMENT EXAMS

One of the most critical areas at the college level that needs rethinking is that of the placement exams used. A large and growing proportion of the students coming in from high schools have been exposed to very different mathematical ideas and emphases, yet mathematics departments continue to assess their ability and knowledge on the basis of a curriculum that is rapidly disappearing. It is little wonder that so many students seem to place lower and lower on these exams despite having had two, three, or four years of secondary mathematics. It may not be that they have failed to learn what they were taught, but rather that they were taught different ideas and methods from those assessed on traditional placement exams. Perhaps it is time to look at both the currently existing secondary curricula and at college-level placement procedures to see if there is any connection between the two. It is quite likely that, at most colleges, many entering students will reflect a traditional curriculum; many other students, however, will reflect a different curriculum. Thus, there will likely be a need for multiple versions of placement tests, one for students with traditional backgrounds and one for students with backgrounds that include coursework based on the NCTM *Standards*.

But this raises a series of additional problems. For example, when teaching a reform calculus course, one of the hardest challenges for many instructors is creating test questions that are in the spirit of the course. The same applies when teaching reform courses prior to calculus. In addition, different instructors bring different perspectives to the types of nontraditional questions they ask; there certainly is no universal consensus about what is "standard" in reform courses. What then should go into a placement exam that measures what students have learned in reform courses? It is relatively easy to compose questions that measure how well students have mastered and retained traditional algebraic skills. It is considerably more daunting to attempt to measure the level of their conceptual understanding, for instance. Just deciding what to measure will be a challenge. In addition, it is questionable to what extent standard placement exams measure a student's readiness for reform calculus or any other reform offering, regardless of the flavor of the student's secondary school mathematics preparation. Therefore, it is becoming increasingly imperative for new placement tests to be developed, either in-house or by the professional test providers.

5. THE ROLE OF TECHNOLOGY

Technology has tremendous implications for the teaching and learning of mathematics, particularly in the precursor courses. The problem educators face is to use the available technology to support the teaching and learning of ematics, rather than as an end in itself.

The use of graphing calculators or computer graphics certainly allows one to luce a much more geometric flavor to almost all mathematics courses. ıctors no longer need to dwell on procedures for producing the graphs of ions, but can instead emphasize the behavior of different classes of functions ocus on questions of scale (i.e., domain and range) to get reasonable pictures highlight the important characteristics of the functions. Rather than devel- g a large variety of techniques designed to solve carefully crafted equations, ıctors can look at the general issue of solving any equation in one variable. If quation is particularly simple, students should be able to solve it using pencil paper. But, if an equation is more complicated, or does not possess a closed- solution, students can "solve" the equation graphically by zooming in on solution point or they can solve it numerically through the table features ıded with most calculators. If a Computer Algebra System (CAS) is available, may be able to solve it algebraically as well.

And, if less time is spent on routine mechanics, it becomes possible to duce other topics. For instance, it is now possible to consider more realistic ications because we are no longer limited to solving equations having simple ed-form solutions. Alternatively, one can introduce a variety of more dern" topics—such as fitting functions to real-world data, which is one of most important applications of mathematics in practice, or recursion and rence equations, which can serve as a preparation for differential equations.

One of the major themes in reform calculus is that of the multiple esentation of the function concept, so that students see and understand the rplay between symbolic, graphical, and numerical approaches, as illustrated in 2. This theme can be even more powerful in the precursor courses, and

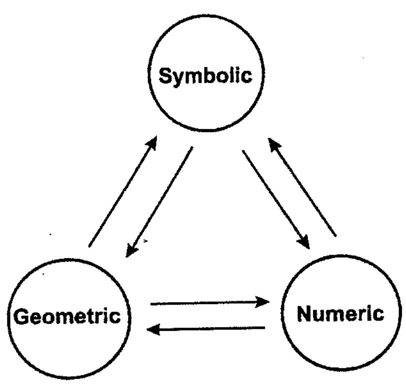

Figure 2. The interplay among symbolic, geometric, and numerical representations.

students should understand how to go from any one of the three representations to the other two perspectives for any function. In the past, mathematics courses were highly focused on going from the symbolic representation of a function to construct its graph manually by first constructing a table of values. In practice, the reverse is by far the more standard problem—given a table of data values or a graphical display, construct a function that models the data. That function can then be used to answer questions that naturally arise in the context as opposed to solving artificially created equations for the sake of practice.

Now that CAS capabilities are available on relatively cheap graphing calculators, there is an even greater challenge for mathematics and a concomitant potential for change affecting the entire mathematics curriculum. Widespread adoption of such a level of technology requires a total rethinking and restructuring of the developmental and college algebra courses to significantly reduce the time spent on pencil-and-paper manipulation skill. In turn, if there is less need to repeat the drill on the same skills from one course to the next, as was previously discussed, then the potential exists to shorten the precursor experience by one or more semesters. However, there are some major drawbacks that could accompany such a move. Students going through such a program might not develop any mathematical maturity, at least not in the senses that mathematicians traditionally recognize. Also, a CAS is useless if one does not use it wisely. There is little doubt that there will be a variety of schools that will experiment with such large-scale programs, with inevitable successes and failures. The mathematics community should pay close attention to the experiences of these experiments, because they may have much to say about the future of the entire mathematics curriculum.

However, even with the availability of such CAS technology, it is evident that students will never be as fast or as accurate as that calculator or computer program—no matter how much instructors drill or motivate their students. Instead, it is essential that faculty adopt a higher goal to produce something far more valuable than an imperfect organic clone of a calculator or computer program.

6. RETHINKING THE PRECURSOR COURSES TO CALCULUS

Mathematics education is currently at a very remarkable stage in history where, because of the confluence of all the pressures and changes, the rare opportunity exists to reassess all undergraduate offerings. Does it make sense to offer a full spectrum of precursor courses, from developmental algebra up through precalculus, that are "clones" of traditional high school courses? That depends on the institution. At schools that have implemented calculus and other reform programs, it makes no sense to take students who have had secondary school courses offered in the same spirit as the reform programs and enroll them ecursor courses that are primarily symbolic manipulation to prepare them for llus courses that no longer treat symbolic manipulation as the primary nasis. But what then of the students who have had traditional secondary ses? How can the college curriculum help them transition to the reform ses?

On the other hand, colleges that offer more traditional calculus courses face y different problem. They may still have to offer standard precursor courses, possibly from a very different point of view. Specifically, it is not necessarily case that students who take such courses are "poor" or "weak" mathema- ly, but just that they have had a nontraditional mathematical background. y shouldn't be viewed as "remedial" students, but rather as students who may d a different approach to the mathematics to bring them up to the desired l.

For that matter, many mathematicians assess students' mathematics ability rely in terms of their facility with symbolic manipulation: those who can do ebra quickly and correctly are "good" students and those who cannot are or" students. But there is much more to mathematical ability. This point ame dramatically clear to the present author a year or so ago during a college ebra class while leading the students to discover and enumerate the behavioral racteristics of cubic polynomials. One of the students, a young woman who s a high school dropout, raised the seemingly innocuous question: "Is it true t every cubic is centered at its point of inflection?" Seeking to draw her out, I ed, "What do you mean by that?" With her eyes screwed up as she tried to ualize her image and with her hands moving in opposing directions, she ponded, "Well, if you start at the point of inflection and move in both ections, don't you trace out the identical path?" This demonstrates an redible level of mathematical insight, especially for someone who had an tremely low view of her own mathematical ability based on teachers' percep- ns and her own performance in traditional secondary courses. Incidentally, I d discovered this lovely fact myself only a year or so before and have since und that many mathematicians do not seem to be aware of this result.

However, based on the reform approach that many students are now getting secondary courses, it is little wonder that they find traditional precursors rribly boring. They have been exposed to much richer mathematical experi- nces—mathematical ideas and substantial (as opposed to artificial) applications d problems. It should come as no surprise that traditional precursor courses are creasingly unsuccessful. It is not that the students are weaker mathematically, ut rather that many have come to expect more from their mathematics courses. If aculty want to develop a traditional level of manipulative skill, they will have to ethink how to motivate students. And that rethinking will likely produce recursor courses that reflect many of the same themes as the reform courses t the secondary level.

In addition, most of these students will come to college having used some technology (usually a graphing calculator) in their secondary mathematics courses. Now that the SAT and AP exams allow graphing calculators, it should be expected that the majority of incoming college students will bring a graphing calculator with them. It seems unreasonable to these students that they are not allowed to use their calculators (or other more sophisticated technology) in college courses. Many secondary mathematics teachers respond to such complaints from their ex-students by advising their current students not to apply to colleges that do not permit use of technology.

7. THE IMPORTANCE OF MANIPULATIVE SKILLS

In today's world, far more people regularly use the concepts of calculus than sophisticated manipulative algebra. People must be able to interpret graphs and tables. They must understand the concept of a functional relationship and how to use it intelligently to make predictions. They need to understand relative growth and decay rates (the real use of percentages); they must know about increasing and decreasing rates of growth (concavity). They must be aware of the notion of accumulation. Many must be aware of the notion of parameters and how changes in parameters affect the behavior of a process. Most need to know something about the modeling process.

But very few people need to factor something like $x^8 - y^8$, let alone $\cos^8 t - \sin^8 t$. Very few must be able to reduce

$$\frac{\left(\dfrac{x_2 - 4x + 3}{x^2 - 5x + 6}\right)}{\left(\dfrac{x^2 + x - 2}{x^2 - 4}\right)}$$

to the obligatory 1. In fact, other than while teaching algebra courses, how many mathematicians ever actually perform algebraic operations as complicated as these? It is easy to picture an academic research mathematician whose teaching load consists exclusively of liberal arts courses or introductory statistics along with virtually any upper-division or graduate course who will never need to use any sophisticated algebraic skills. Equally important, much the same can be said about most engineers and scientists. It will be increasingly true in the future as both their training and their practice of engineering and science depend more heavily on technology.

This certainly does not mean that there is no longer a need to teach algebra and that students should not be expected to perform any algebraic operations whatsoever. However, there likely will be a very different balance. Certain raic techniques will continue to be taught and, if anything, are likely to be ıasized more than in the past. For instance, operations with properties of ɿents and logarithms fall into this category because they are needed to solve ' of the problems that are receiving greater emphasis. But many other skills, ɔularly those associated with solving equations, will receive far less atten- For instance, it is less important to be able to factor a polynomial to find its if one can locate any real root to any desired degree of accuracy by ɿical or numerical means. Similarly, applying an inverse trigonometric ɿion finds just one possible solution to a trigonometric equation, but all ions can be found graphically. Also, it is far more time-consuming (as well niting in terms of the degree of complexity expected of students) to solve a ɜm of linear equations by hand when it can be solved by converting it to a ɔr–matrix equation and pushing the appropriate keys on the calculator; this ation simultaneously provides an exposure to the power of matrix algebra and pplications.

A redefinition of the word *algebra* is now needed, one that has far-reaching ications for the courses that teach it. Algebra should be viewed as far more just a collection of manipulative tools for moving symbols around and for ing carefully constructed equations. This is especially true in today's fast- ıging world. The traditional precursor courses were designed primarily to ɜlop algebraic skills that once were essential for success in later courses. The e availability of technology and changing requirements, especially in the nt disciplines, requires a rethinking of this paradigm. For instance, students in er-division courses in engineering and the sciences do relatively little pencil- -paper mathematics; instead, they focus on developing mathematical models lescribe real-world phenomena. These models typically involve differential or erence equations, matrix methods, or often probabilistic simulations. The lents examine the behavior of the solutions, particularly as the parameters ɔrlying the phenomena change. Similarly, students in business, the social ɜnces, and the biological sciences are expected to recognize trends from sets of a, construct appropriate mathematical models to fit the data, and make responding predictions based on the models developed. This is actually ɿarkably similar to what students in lab courses have been doing for centuries; difference is that the students in the business and social science courses ıically use spreadsheets for the analysis rather than hand-drawn graphs.

Unfortunately, in the minds of many students, there is little connection ween what they see in mathematics classes, where the problems look like: *Find equation of the line through P* (1,2) *and Q* (4,8), and what they encounter in ɜir other courses, where the problems are more like: *Given a set of points, draw line that best fits the points, find its equation, and answer the following estions about the situation from which the data came.* Faculty in other antitative disciplines complain that the mathematics taught is too abstract.

Mathematicians tend to interpret this as a complaint about too much rigor: too much definition–theorem–proof structure; too much delta–epsilon, and so forth. In reality, what the other faculty are communicating is that the methods used in mathematics courses are context-free and idealized to the level of sterility. Students do not recognize that they need to apply the same ideas and techniques to problems that vary even slightly from the textbook examples. For example, when the students see situations where the letters used for the variables are other than x and y or the numbers used are not one-digit integers, they often have no idea that there is any connection at all with mathematics.

In general, the overarching emphasis on algebraic manipulation in traditional precursor mathematics classes does not provide the foundation that students need for these disciplines, nor does it adequately prepare them for reform calculus. Instead, a broader preparation is needed, one that better reflects the practice of mathematics. Students need to learn how to

1. Identify the mathematical components of a situation (i.e., model it)
2. Select the right tool (e.g. paper and pencil, graphing calculator, CAS package, spreadsheet) to solve the problem
3. Interpret the solution in terms of the original situation and, if necessary, change the assumptions used in the model (i.e., introduce additional factors)
4. Communicate the solution

The focus of much of traditional mathematics education, though, has been to emphasize one particular set of tools (algebraic manipulation) with little or no emphasis on any other tools or any other part of this process. However, it is certain that no one can anticipate the kinds of tools that will be available 10 years from now. Certainly, any routine procedure that is important has already been programmed and is available at the push of a button. If solving routine template problems continues to be a primary emphasis of mathematics courses, the education of our students will be outmoded before they leave the classroom. It is no longer defensible to offer courses in which the primary focus is having students practice one set of routine problems after another and then have them reproduce those solutions on an exam. In the past, one might have argued that this was unavoidable in order to develop the high algebraic skill level needed for calculus; today, that argument may no longer be valid.

8. RESISTANCE TO CHANGE

There are a variety of reasons for resistance to change in the precursor courses. Many are the same as those seen with calculus reform, although there are

; that are unique to the courses below calculus. Consider the following nges and possible solutions.

enge: For some, the resistance is purely philosophical. They feel that reform courses are changing the very nature of the subject. Many such individuals, particularly those who do not actually teach any of the courses, have a strong suspicion that reform is just a euphemism for watering down the courses.

ion: This kind of resistance can often be overcome by showing the greater depth of mathematics being covered in the new courses and especially by demonstrating the deeper level of student understanding and achievement.

lenge: For some, the key argument against reform is: "That worked for me and everyone else I know, so it is the only way to produce more of me."

ion: The people who espouse such views are often new faculty members recently out of graduate school. They often just need a year or two in front of a classroom to learn quickly that the overwhelming majority of students are not interested in becoming mathematicians. In the past, there really was little in the way of an alternative to the traditional curriculum. However, with the options available today, a little support and guidance can often convince these faculty that curricular change can improve student learning, as well as the environment in their classes.

llenge: Some fear, often with good reason, that something will be lost in the reform courses. If an instructor introduces new material or new emphases or has the students working in groups in class, it will be impossible to *cover* everything.

tion: In the past, topics and methods once considered essential were eliminated from the curriculum—though at a much slower pace than is happening today. For example, look at what happened to the Law of Tangents.

On the other hand, it is reasonable to expect that a student who has learned some mathematics well at any given level of the curriculum and who has also learned how to think mathematically should be able to later pick up that textbook and independently learn something. For instance, if partial fractions are not taught in precalculus, a student with a solid precalculus-level background should be able to learn a little about partial fraction decompositions while taking Calculus II. Similarly, a student who has not been exposed to integration of rational functions via partial fractions in calculus should be able to read up on it quickly while taking a course

in differential equations when using partial fractions to derive Laplace transforms. Of course, this freedom to change content within courses as appropriate requires open and frequent communication among instructors.

Challenge: Some mathematicians are cautious and uncomfortable with technology. They often have not fully thought through the positive implications for teaching mathematics. They may be concerned that students will ask questions that they are not prepared to answer, either about the technology itself or the different mathematical questions that can arise. Some may fear that the technology will replace the mathematics as the primary focus of the course.

Solution: Most such objections can be overcome by exposing these individuals to the technology in a supportive environment of faculty training workshops. However, the use of technology cannot be forced on instructors as something that must be used in class; their input and feedback must be part of the process of change.

Challenge: At many schools, the faculty teaching the precursor courses are people with relatively limited mathematical backgrounds and/or experience. They may be part-time faculty or graduate assistants, or individuals who have a very high comfort level with what they know and have been teaching for years, but do not have the personal confidence to try something new and different. Such individuals are often tentative about adopting renewal ideas because they lack knowledge about some of the mathematical subject matter. For instance, the notion of families of functions or slope fields or regression and correlation analysis may not have been part of their mathematical training.

Solution: Often, the most effective approach is to offer on-going training workshops, though local circumstances may well make it difficult to ensure that part-time faculty will be able to participate. Unfortunately, it is not always easy to secure the funds needed to pay for such participation. Nevertheless, it is worth the effort to explain to college administrators that mathematics is changing—and why—and request additional funding for faculty development. Evidence of improved student performance and retention, as well as closer ties with other quantitative subjects, is an important part of such a conversation.

Challenge: At some schools, the precursor courses are the responsibility of separate developmental departments and the faculty in these departments may not be aware of the changes that are taking place in calculus and other courses in the mathematics department. They likely will see their role as unchanging, even though the students in calculus are being held to a different standard and are having very different experiences in class, in their homework, and on their exams.

tion: In such situations, there is an urgent need for on-going communication to acquaint these instructors with the new emphases in calculus and the changing skills that students now need.

llenge: Many faculty, particularly at two-year institutions, carry very heavy teaching loads. In addition, reform courses are very labor and time intensive for the instructor. Nonroutine conceptual or applied problems can be difficult to construct; they are also more time-consuming to grade on exams or homework. Similarly, grading extensive written assignments and reports, as well as mentoring the student projects leading to such reports, is also time-consuming. It is little wonder that faculty are resistant to implementing reform courses, even when they agree philosophically with the goals of the courses.

ution: A support system involving all members of a department can help make the transition easier. Often, a feeling of camaraderie can make the extra effort less burdensome and the opportunity to exchange ideas with colleagues can be a tremendous stimulus in developing new problems and examples. It is also helpful to include frequent reminders that the efforts are in the long-term best interests of the students. Finally, it may be possible to convince administrators to provide financial support and/or release time if a sufficiently strong case can be made that the extra time involved presents an overwhelming obstacle. If the rationale for introducing the new approach is compelling (say, the expectation of having a better success rate), it may well be a convincing argument.

allenge: Faculty at two-year institutions are legitimately concerned about the transferability of reform courses.

lution: According to *Assessing Calculus Reform Efforts*,[5] a significantly higher percentage of four year colleges and universities have been implementing calculus reform efforts than two year colleges. In particular, approximately 76% of all doctorate-granting institutions, about 74% of all master's-granting institutions, and 74% of all bachelor's-granting institutions reported having implemented either moderate- or large-scale calculus reform. In comparison, about 54% of two-year colleges reported such activities. Furthermore, almost all of the institutions that reported such efforts indicated that they anticipated larger-scale reform activities in subsequent years. Thus, it appears that transferability is likely not an issue at all. In fact, if an instructor at a two-year college does not teach in a reform style, the

students may well be at a disadvantage when they transfer to a university that likely utilizes such techniques.

Challenge: One argument occasionally raised against reform is that students will not be adequately prepared for physics or chemistry or "X."

Solution: The changes that are taking place in the precursor courses, as well as in the reform calculus classes, very closely mirror comparable changes that are taking place in fields such as engineering, physics, and chemistry. These changes include a greater emphasis on conceptual understanding, a lessened emphasis on algebraic manipulation, group work on projects, the use of technology, and so forth (see Gordon[6]). While they may not be taking place in every department in every college, they are certainly happening with increasing frequency—especially in engineering departments that are accountable to the newly developed standards of the national Accreditation Board for Engineering and Technology (ABET). The faculty in other departments that are renewing their own courses are important colleagues with whom mathematicians should collaborate.

9. WHAT A MODERN PRECURSOR COURSE SHOULD EMPHASIZE: SUMMARY RECOMMENDATIONS

A good precursor course is one that takes a broader perspective than just preparing students for calculus, whether it is reform calculus or traditional calculus. The reality is that the overwhelming majority of the students will not go on to calculus and the precursor course must deliver something useful and important to everyone. The challenge is to provide a solid preparation for calculus while simultaneously meeting the long-term needs of all students. What then should such a course feature?

9.1. Students Must Understand Variables and Functions

- Students must see variables as representing the values of quantities and, in order to meet the needs of other disciplines, the letters used for variables should include more than just x and y.
- Students must achieve a solid understanding of the difference between independent and dependent variables.
- Students must see functions through multiple representations: as formulas, as graphs, as tables, as verbal depictions, and as dynamic processes that describe realistic phenomena; they must understand the perspective represented by each.

Students must come to understand the practical limitations of any given function, in the sense of domain and range. In practice, domain and range mean far more than just looking for those points where one divides by zero or where one takes the square root of a negative number. For instance, if one creates a linear model for the world record times in the mile run based on data from the start of the twentieth century, it makes little sense to extend it backward to predict how fast Columbus or Plato could have run that distance.

Students must recognize certain classes of functions both from formulas and from graphs; specifically, linear, exponential, power, polynomial, and sinusoidal functions. When a student sees a set of data, a chart, or a graph, he or she should be able to match it with functions that behave in the same manner.

Students must understand fundamental ideas about the behavior of functions, such as growth versus decay, increasing versus decreasing, and concave up versus concave down.

Students must understand the effect of various parameters on the behavior of functions. For instance, the slope of a line must be interpreted in the context of units of measure and scale, not just so many boxes over so many boxes.

Students must understand the effects of transformations—stretching and shifting—on functions.

pplications Should Be a Unifying Theme for All Precursor Courses

ıe majority of students taking precursor courses are far more interested in ›lications of mathematics than they are in the mathematics itself. Instruc- ‿ıld capitalize on this interest and, in the process of developing an iation for the power of mathematics, perhaps an appreciation for the beauty mathematics will follow. It is also possible to incorporate a tremendous of mathematical thought through applications. In particular:

Applications should be used to illustrate connections between the mathematics and the real world, connections with other courses, and (perhaps most importantly) connections to students' personal experiences.

Applications—both the contexts and the mathematical models used—should be realistic instead of artificial expressions that are created only to have the answers work out "nicely."

Applications should help students to see how real problems lead to equations. This realization should then be used to help students better

understand the meaning of a "solution."

- Applications should be used to help students explain mathematical ideas and solutions in everyday language, both oral and written. From a mathematical point of view, a solid explanation is the first step toward a formal proof.
- Students should be exposed to nonroutine problems that require applying mathematics to new situations.
- Applications of the mathematics should be used as motivation for students to take additional courses in mathematics.

9.3. Students Should See the "Big Picture"

Even the best students will take only a few ideas and methods with them when they leave a course. Instructors need to recognize this fact and make sure that students understand the fundamental concepts and methods of the course.

- Students should not be overwhelmed by too many specific examples at the expense of the main ideas.
- Students should see how certain fundamental ideas arise constantly throughout mathematics, not just during a unit on the topic. For example, exponential behavior is extremely common in real-world phenomena; linear behavior is certainly the most common. Therefore, exponential functions should not be approached as a topic to be covered in one or two weeks, with no connections to the rest of a course. Similarly, systems of linear equations should be seen as a unifying notion, not just a seemingly minor topic covered for a week or so.
- Students should spend less time on the mechanics of finding solutions for such equations and use this additional time learning to interpret such solutions.

9.4. Students Should Become Familiar with a Variety of Mathematical Tools—Symbolic, Graphical, and Numerical—and the Technology that Allows Us To Apply These Tools

Technology is everywhere and certainly will become far more prevalent in the future. The use of some type of technology, whether a graphing calculator, a spreadsheet, or a computer algebra package, should be an essential component of any modern precursor course.

Students should become comfortable with all of the different tools for solving equations. For example, any equation involving a single variable can be solved using graphical or numerical methods to any desired degree of accuracy;

ı particularly simple kinds of equations can be solved exactly using ›lic methods.

ₖethinking the precursor courses to reflect these principles and implement- .e necessary changes may well appear as a daunting challenge to many. ver, the rewards—both for the students and for the instructor—are well the effort. As the reform movement continues to influence more of the :matics curriculum, numerous textbooks and project materials are becoming ble that embody these principles. Simultaneously, the needs of faculty ring to teach from these texts are being addressed for the first time. :rous faculty development workshops are being offered by professional izations, by publishers, and by the individuals involved in the projects.

In addition, many written reports and professional presentations describe idual and departmentwide experiences with such courses. These reports de guidance for implementing such programs and can help mathematicians others understand that these changes to the precursor courses provide a ıe opportunity to improve the mathematics education of students.

CONCLUDING REMARKS

There is an old Chinese curse that says: You should live in interesting times. ;e in mathematics education likely feel that they have been hit with a multiple nmy of that curse. And, the reform movement is, if anything, merely ering steam. Mathematicians will face increasing pressures for greater ges in the coming years.

Most of the changes discussed describe things that are happening at the ege level, though they certainly have major implications for how secondary ol students are prepared for college mathematics. At the same time, parable changes are being implemented across the secondary curriculum as NCTM *Standards* are more widely adopted and as traditional textbooks are aced with more reform oriented textbooks. Thus, the colleges can expect to increasing numbers of students with very different mathematical experiences proficiency than previously. Students will have had less formal algebraic ipulation, but a greater emphasis on conceptual understanding and multiple esentations of those concepts; consistent exposure to the use of technology, ost entirely in the form of graphing calculators; more experience in writing communicating mathematics; more experience with extended, realistic lications of the mathematics; and more experience in learning in nontradiıal classroom settings and with more varied methods of assessment used.

This, in turn, will put ever increasing pressures on those college faculty who e been resistant to curricular change in the hope that it will either fade away or t they can resist until retirement. As mentioned above, there are certainly many

who have resisted on legitimate philosophical or other grounds. In the past, there has been a seamless transition between secondary and college mathematics because all schools (secondary and college level) offered the same courses in the same spirit. The reform efforts at all levels will soon create a very different—though equally seamless—transition between secondary school and college. But in the process, mathematicians should all expect a lot of bumps as the pieces rub against each other to generate the kind of perfect fit that they would like.

What also is clear is that the mathematics students coming out of this new curriculum model will be different. Instead of pencil and paper, these students will naturally turn first to some kind of graphical tool to investigate what a process looks like or whether an idea makes sense. They then will turn to the algebraic techniques only for general verification. Certainly, the next generation of mathematicians will not be clones of the current one; they will bring a very different vision to mathematics and related areas.

But what about the relatively few people—mathematicians, physicists, and computer scientists, for example—who will need to know the full array of symbolic operations? It now seems unreasonable to have everyone learn those skills, particularly as prerequisite skills to courses that may no longer require them. It is clear that this did not work well for the overwhelming majority of students, even when such skills were essential for success in calculus and related courses. Now that such skills are less necessary, there is no reason to expect all students to master them.

It may therefore be reasonable, in the not-too-distant future, to see junior- and senior-level college courses in advanced manipulative algebra offered at large research universities for the few students who really need it, while the overwhelming majority of students are exposed to more fundamental mathematical reasoning, mathematical ideas, and realistic applications.

It is evident that the secondary school mathematics plate has shifted dramatically and will continue to shift even further. The postsecondary mathematics plate may be shifting in the same directions, so that at some schools, there will continue to be a relatively smooth transition. However, many other schools may want to recall what occurs when the earth's plates shift in different directions; and an earthquake is not a pleasant experience.

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CHAPTER 7

Program Evaluation and Undergraduate Mathematics Renewal

The Impact of Calculus Reform on Student Performance in Subsequent Courses

Jack Bookman

Educational research and research with human subjects must, by its nature, be flawed—at least in comparison with the standards set by scientific research. Program evaluation, in particular, must deal with complex and difficult-to-control situations. Therefore, conclusions from such research must be tentative and qualified. Some argue that because of this complexity and ambiguity, program evaluation is not worth doing. However, program evaluation—flawed as it is—can provide valuable insight into what a program has accomplished and what components have contributed to or impeded its success.[1] This chapter will address a particular aspect of the evaluation of current calculus reform efforts, namely, the effect of calculus reform on student performance in subsequent courses. It will summarize the findings from several studies on this subject and will address the corresponding methodological difficulties.