Report of the Content Workshop

Introduction

This report addresses the philosophy and content of the course or courses that should serve as the preparation for a reform calculus course. As such, we do not explicitly attempt to discuss the preparation for other college level mathematics courses, such as discrete mathematics, statistics, and finite mathematics. As a result, the ideas that follow are focused on preparing students for calculus at the current place and time. The recommendations below both affirm the changes proposed by the NCTM *Standards* and represent a transition from high school mathematics to the new calculus courses.

The types of courses that we address are offered in high schools, two-year colleges, four-year colleges and universities. The students at different institutions vary in terms of level of mathematical maturity, mathematical preparation and skills, ability and motivation. There is great variation in the time allocated to the preparation for calculus: from several years in high school to as little as one semester in college. We have not tried to distinguish between a single "precalculus" course and the sequence of courses from algebra through precalculus that represents the full preparation for calculus. Consequently, individual recommendations may not necessarily apply to all institutions, but the overall principles enunciated should apply to all mathematics offerings.

Finally, what we have addressed here is preparation for reform calculus, not necessarily for traditional calculus. However, we believe that many of the recommendations apply to the preparation for traditional courses as well.

What are the content goals of the "new" calculus courses?

The new calculus reform projects involve a change in focus in what it is important to learn, what is taught, how it should be learned, and the role of technology in the learning process. While there are many different versions of reform calculus, most of them share many essential elements:

- The emphasis of calculus should be on the fundamental concepts of the subject, not on symbolic manipulation. In some of the courses, the traditional manipulation is relegated to work that is done by a computer algebra system such as *Derive* or *Mathematica*; in others, the level of manipulation is reduced and replaced by an increased emphasis on conceptual understanding of the fundamental concepts of calculus and their applications. In none of the reform calculus projects is the emphasis on having students perform long lists of similar exercises to gain dexterity at finding limits, derivatives or integrals or to produce graphs of functions.
- The topics in calculus should be approached symbolically, graphically, numerically and in verbal and written form.
- Calculus courses should emphasize modeling the real world and providing experience with problem-solving.
- Appropriate technology should be available at all times for graphing, numerical computations, and symbolic manipulations.
- The new calculus courses place a different level of expectation on the part of the students. Students are expected to think and not just to perform routine operations. This expectation is reflected in exercises, projects, examinations, and written assignments.

Principles for the "New Precalculus Courses"

The principles enumerated above for the reform calculus courses have immediate implications for the course or courses that prepare students for calculus. If the level of manipulative skill required for calculus is significantly reduced, there is less need to develop those skills in the precursor courses. The time saved can be devoted to other topics or to other emphases that better reflect what will happen in calculus. If there is a greater emphasis in reform calculus on conceptual understanding and mathematical thinking, then we should prepare the students by emphasizing these in the precursor courses.

Several new precalculus projects are currently underway, primarily in the secondary schools. These projects also involve less paper-and-pencil manipulative algebraic and trigonometric work on the part of the students. As with their calculus counterparts, some of these projects have students perform very little algebraic manipulation; others cover the gamut of ideas, but do not have the students perform the full array of complex manipulations usually covered in traditional courses. In none of these projects are students expected to perform long sets of similar exercises for the express purpose of developing high levels of manipulative skills.

The Content Issues Workshop participants believe that most of the points made above for the new calculus should apply equally to the courses that prepare students for it. We also believe that, in the process of implementing these principles, many different versions of precalculus courses will emerge, much as many different versions of reform calculus courses have emerged. We feel that this diversity will be healthy as different philosophies and organizations of topics will reflect a variety of perspectives.

In general, the participants agreed on the following set of fundamental principles:

Courses designed to prepare students for the new calculus should:

- cover fewer topics, but each topic should be covered more thoroughly and with more emphasis on fundamental concepts.
- place less emphasis on complex manipulative skills.
- teach students to think and reason mathematically, not just to perform routine operations. This higher level of expectation should be reflected in exercises, projects, examinations, and written assignments.
- approach each topic symbolically, graphically, numerically, and in verbal and written form with the aim of helping students construct appropriate mental representations of mathematical concepts.
- emphasize modeling the real world and develop problem-solving skills.
- make use of all appropriate calculator and computer technologies for graphing, numerical computations and symbolic manipulation. The full power of technology should be introduced in the service of learning the mathematics.
- promote experimentation and conjecturing.
- provide a solid foundation in mathematics that prepares students to read and learn mathematical material at a comparable level on their own, and to apply what has been learned in new situations.
- simultaneously serve mathematics and the physical sciences, the biological sciences, the social sciences, and other fields. The mathematics included should be presented as an elegant, unified and powerful subject that describes processes in all of these areas.

Rationale for the Precalculus Principles

There was complete agreement that traditional precalculus courses focus far too heavily on the process of learning procedures and far too lightly on learning mathematical concepts and their applications. They also attempt to cover too many topics, often as a consequence of the focus on algebraic techniques. Thus, the workshop consensus is that the new courses should de-emphasize complex manipulation, should cover fewer topics, but should cover those topics more thoroughly and with more emphasis on fundamental mathematical concepts.

In the process, there was considerable discussion, both during the formal working group sessions and in informal conversation, regarding the extent to which these courses should de-emphasize manipulation. While no one spoke in favor of long lists of routine exercises, there was no agreement on how much time could be saved or, equivalently, what should be deleted from the arsenal of paper-and-pencil techniques that good students should now be expected to possess. The high school teachers in the group pointed out that the NCTM *Standards* already call for a move in this direction and that students coming out of the high schools beginning in the next few years will likely have a lower level of manipulative skill than universities presently expect of incoming students.

The manipulative skills most often discussed were factoring, operations with rational polynomial expressions, radical and radical equations, and trigonometric identities. Any consideration of the issues, however, must include simplifications and the solving of equations, inequalities and systems.

The issue is enormously complex. Consider, for example, the problem of reducing an expression such as

$$\frac{\left(\frac{x^2+x-2}{x^2-3x+2}\right)}{\left(\frac{x^2+3x+2}{x^2-x-2}\right)}$$

The solution of this problem is almost exclusively an exercise in factoring. Students who do not see such problems will likely not be handicapped in any of the new calculus courses; many students who are forced to perform such manipulations are likely to be discouraged, one way or another, from ever going on to calculus. Further, as some of the high school teachers pointed out, the traditional full treatment of factoring takes at least three months out of their syllabus; much more mathematics could be done if a significant portion of that time could be saved.

Alternatively, consider

$$\frac{1}{x-1} + \frac{1}{x+2}.$$

For some, this was the kind of manipulation students should not be expected to perform by hand; the ability to perform it by machine and to check the results numerically using substitution would suffice. Others felt that students should be able to perform this addition by hand and employ similar techniques whenever decomposition into partial fractions was required. However, no one spoke in favor of students needing to combine, by hand, significantly more complicated expressions involving, for example, quadratic terms in the denominator.

As a third example, there was a question of how many trigonometric identities should be required. Some felt that the Pythagorean identity $\cos^2 x + \sin^2 x = 1$ would suffice. Others would be content with this and the addition formula for $\cos(x + y)$. Others felt that students should know these as well as a handful of additional formulas. No one believed that students should have to learn the full arsenal of identities, such as the formulas for products of trigonometric functions, that are in traditional courses.

In a related direction, some participants felt that students should have some experience proving trig identities analytically, though certainly not to the extent currently favored. Others did not believe that this was essential. Yet everyone felt that students should be able to check whether or not a statement was an identity, either numerically by substituting values for the variable, or graphically by comparing the graphs of the two supposedly equivalent expressions.

As a fourth example, no one felt that it was necessary to require students to solve radical equations with two radicals. The only time that this comes up subsequently is in the derivation of the formulas for the ellipse and hyperbola and that can be demonstrated by the instructor. Yet, everyone felt there is value is solving relatively simple radical equations with a single radical term as an illustration of the use of an inverse function.

In summary, the resolution of these questions was felt to be beyond the scope of a brief conference such as this one. Some felt that research into the precise manipulative skills required in the new calculus courses should be the first step. Others felt that opinions on this issue are not likely to be swayed by research of any kind and that only time would tell which manipulative skills will survive. Some felt that the proof would be in the new courses that are developed to prepare for the new calculus.

There was complete agreement that conceptual understanding is central and must not be lost in the process of learning procedures. Thus, a major focus for the new precalculus courses should be to teach students to think and reason mathematically, not just to perform routine operations. This higher level of expectation should be reflected in exercises, projects, examinations, and written assignments.

In order to achieve this higher level of conceptual understanding, the participants agreed that topics should be approached symbolically, graphically, numerically, and in verbal and written form with the aim of helping students construct appropriate mental representations of mathematical concepts. The consensus was that nonalgebraic ways of understanding mathematics should be highlighted, and the meaning of each representation, be it algebraic, visual or numeric, should be emphasized throughout. Thus, it is critical that students learn to investigate symbolic problems graphically and vice versa and that numerical values should be used consistently. Further, the participants agree that we should emphasize to students how important it is to develop the judgment to choose the correct tool (technology or pencil-and-paper). Finally, the courses should emphasize connections among different mathematical ideas.

Another common thread in the discussions was the need to make these courses problem-driven in the sense of using the applications of mathematics as the motivating idea. Thus, the courses should emphasize modeling the real world and develop problem-solving skills. The applications chosen should be of interest to students, not just to the instructor or to experts in another discipline. Finally, the applications chosen should relate students' real-life experience to mathematics.

Much of the discussion on the need for new preparation for calculus hinged on the issue of technology. It has changed what should be taught and how it should be taught. Several participants described a new calculator that has 64K of memory (more than the original Apple II), a full implementation of a computer algebra system, and two expansion slots that each take a 2 meg card. While many in mathematics think that technology affects only them, consider the impact of an expansion card with a full interactive dictionary/thesaurus on the English curriculum; now that calculators are allowed on the SAT exam, think what this would do to the verbal part; consider that the 2 meg card could also contain all related verbal questions from previous SAT exams.

It was evident that no participant felt that technology should or could be ignored. Instead, the consensus was that the new courses should make use of all appropriate technology for graphing, numerical computations and symbolic manipulation. The only caution expressed was that technology should not become the focus of the course, but rather that it should still be a mathematics course. Thus, the full power of technology should be introduced in the service of learning the mathematics. The extent to which this is done is a matter of individual philosophy. As with the reform calculus projects, some efforts will undoubtedly focus heavily on technology; others will utilize technology in less evident ways.

One of the primary uses envisioned for technology in these courses is to promote experimentation and conjecturing. Technology should not be used merely to get answers. Rather, students should use it to explore mathematical ideas without becoming overwhelmed with the manipulative work. For example, what patterns could they discover among the resulting terms by expanding $\sin 2x$, $\sin 3x$, $\sin 4x, \ldots$ using a computer algebra system?

Another area discussed briefly by the participants was whether the new courses should be narrowly focused on preparing students for calculus or should be broader to prepare them for mathematics in general. While we agreed to adopt the former in order to meet our charge for the workshop, it was evident that most participants welcomed the idea of broadening the content considerably. Suggestions to include additional topics such as data analysis, probability and simulation, recursion and iteration, and matrix algebra and its applications arose repeatedly. Some participants indicated that such topics might even be introduced in these courses in the service of preparing students for calculus.

Finally, some general philosophy regarding the new courses emerged. The participants believed that these courses should provide a solid foundation in mathematics that prepares students to read and learn mathematical material at a comparable level on their own, and to apply what has been learned in new situations. In particular, they felt that the principle of covering fewer topics means that different students will undoubtedly have "missed" some topic that may surface in a subsequent course. However, if the preparatory course prepares them to learn independently, they should be able to pick up a textbook written at the comparable level and learn some mathematics on their own. They should also be able to carry the mathematical knowledge and cognitive processes they have learned in one course over into other courses, both in mathematics and in the client disciplines.

Further, the participants felt that the new courses should simultaneously serve mathematics and the physical sciences, the biological sciences, the social sciences and other fields. The mathematics should be presented as an elegant, unified and powerful subject that describes processes in all of these areas.

Finally, the participants believed that every course should be approached as if it were a terminal course, not merely as a course that is intended to prepare students for a subsequent course. The result will be far better courses that provide students with a truly enriching mathematical experience.

Some Specific Ideas and Suggestions

The following are some more specific ideas and suggestions that arose during the workshop deliberations on particular topics and points that should be included in designing and implementing reform precalculus courses.

1. The Role of Algebraic Manipulation
   (a) Reduce the level of complex manipulation throughout.
2. Geometry and Graphs
   (a) Areas, volumes and geometric figures
   (b) Windows and scale
   (c) End behavior, local versus global behavior of functions
   (d) The idea of the "complete" graph of a function
   (e) Some graphs must be done by hand
   (f) The effect of parameters on scale
3. Numerical Work
   (a) Number sense
   (b) Use "messy" numbers; real data
   (c) Estimation and approximation
   (d) Accuracy and significant digits
   (e) When do we need an exact answer?

4. Thinking and Reasoning
   (a) Experimentation and conjecture leading to the writing of insightful proofs (this should not be a major focus, but an on-going theme)
   (b) Graphical verification (e.g., algebraic or trigonometric identities)
   (c) Use of quantifiers
   (d) Distinguish among conjecture, proof, counterexample, hypothesis and conclusion
5. A Continuing Theme: Functions
   (a) Functions as Modeling the World
   (b) Functions given by data/graphs and formulas
      i. Linear functions, constant rate of change
      ii. Exponential functions, constant ratio; for example, the population of California, 1890–1960
      iii. Trigonometric functions as model of periodic phenomena; for example, time of sunrise as a function of date
      iv. Other applications discussed: projectile motion, stopping distance for a car
   (c) Mathematical Classes of Functions will probably include:
      i. Linear
      ii. Exponential (any base) (more emphasis than in traditional courses)
      iii. Logarithms
      Note: exponential and logarithmic functions should be used throughout the course.
      iv. Periodic functions
      v. Polynomial functions (less emphasis than before)
      vi. Rational functions (much less emphasis than before)
   (d) Characteristics of Functions that should be included wherever appropriate
      i. Zeros
      ii. Rates of change
      iii. Increasing, decreasing, concavity
      iv. Extrema
      v. Composition
      vi. Inverses (Note: inverse problems should be used throughout as a unifying principle)
6. Other Topics that might be included:
   (a) Probability and simulation
   (b) Data analysis
   (c) Recursion, iteration, sequences and limits
   (d) Algorithmic thinking
   (e) Matrices and systems of linear equations
   (f) Combinatorics and counting

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