WHEN DO POLYNOMIALS LOOK LIKE CIRCLES? AN EXPLORATION WITH PARAMETRIC CURVES

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The answer to the question posed in the title is: When their graphs look like one of the two curves shown in Figure 1. Of course, this leads to the follow-up questions, Which is the circle?, and Which is the polynomial? A more significant question is: How can a polynomial have such a graph?

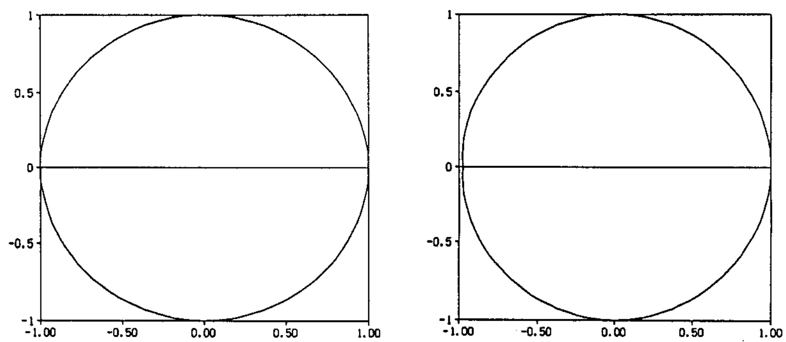

Figure 1: A polynomial in parametric form and a circle.

In a recent article [1], the author explored the idea of applying Taylor polynomial approximations to curves in polar coordinates. A natural extension is to conduct a similar investigation of Taylor approximations for parametric functions. Suppose we have the parametric representation

$$x = f(t),$$
$$y = g(t)$$

for some curve in the plane and consider the associated Taylor polynomial approximations

$$x = P_n(t) \approx f(t),$$
$$y = Q_m(t) \approx g(t)$$

where the approximating polynomials are of degree n and m respectively.

The resulting curve can be thought of a polynomial graph, though in actuality, when we eliminate the parameter t, the result is typically some non-trivial plane curve. In particular, the curve on the left of Figure 1 is the graph of

$$x = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \frac{t^8}{8!},$$
$$y = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \frac{t^9}{9!}$$

for t between $-\pi$ and π while the circle to the right is the graph of

$$x = \cos t,$$
$$y = \sin t.$$

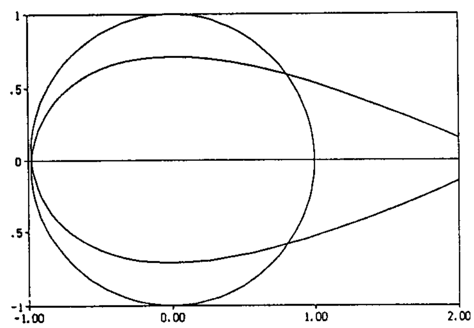

Figure 2: The polynomial on a larger interval.

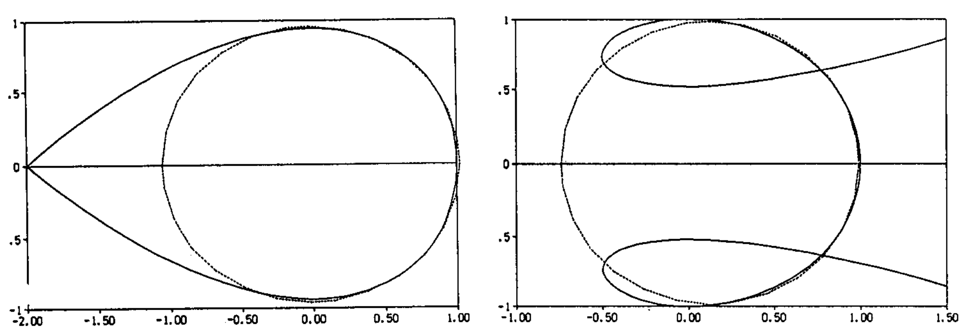

Figure 3: The approximations (a) $\{P_2(t), Q_3(t)\}$, (b) $\{P_4(t), Q_5(t)\}$

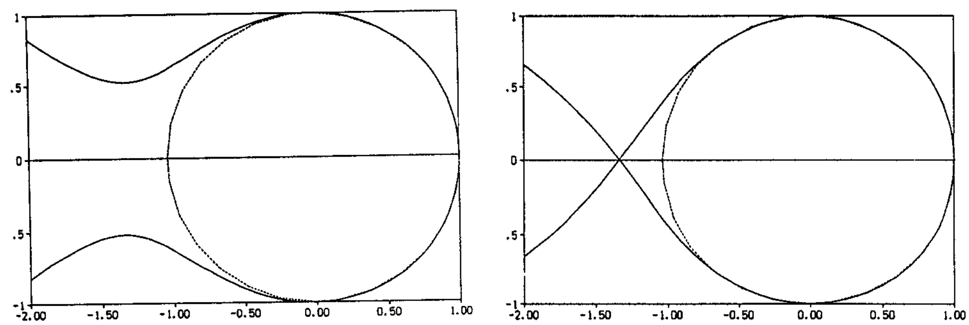

Figure 3: (c) $\{P_6(t), Q_5(t)\}$, and (d) $\{P_6(t), Q_7(t)\}$ on the interval [-4, 4].

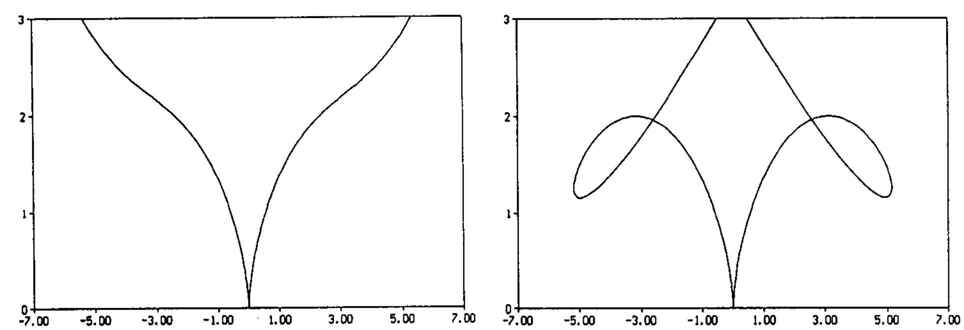

Figure 4: The approximations (a) $\{P_7(t), Q_6(t)\}$ and (b) $\{P_9(t), Q_{10}(t)\}$

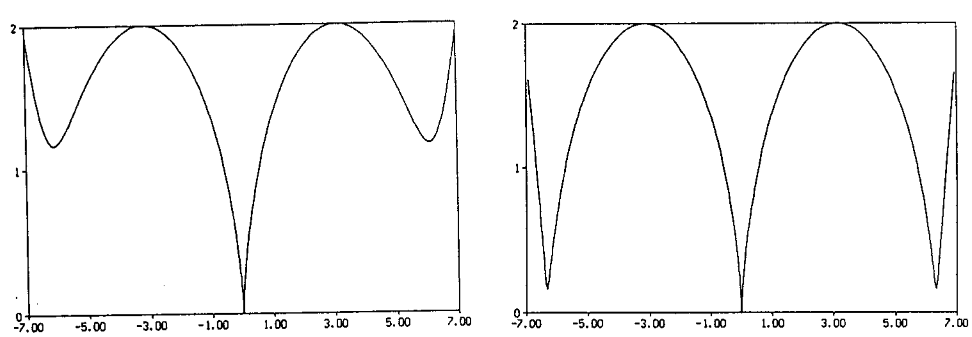

Figure 4: (c) $\{P_{11}(t), Q_{10}(t)\}$, (d) $\{P_{15}(t), Q_{14}(t)\}$ to the cycloid on the interval [-7, 7].

If we extend the interval to [-5, 5], say, then the resulting "polynomial" graph is as shown in Figure 2 where we have superimposed the graph of the circle for comparison in judging the effectiveness of the approximation.

We begin with some fairly evident observations about the behavior of parametric curves given by polynomials: $x = P(t), y = Q(t)$.

1. Since $P(t)$ and $Q(t)$ are defined for all real t, the curves must be unbounded.
2. The roots of $P(t)$ are the y-intercepts of the curve.
3. The roots of $Q(t)$ are the x-intercepts of the curve.
4. The local maxima and minima of $P(t)$ correspond to vertical turning points of the curve.
5. The local maxima and minima of $Q(t)$ correspond to horizontal turning points of the curve.
6. The parametric curve crosses itself at "symmetric" points (x_1, y_1) and (x_2, y_2); that is, if $P(t)$ and $Q(t)$ are plotted as functions of t on the same axes, then the symmetric points $(t_1, x_1), (t_1, y_1), (t_2, x_2)$ and (t_2, y_2) form the corners of a rectangle.
7. The slope of the tangent line to the parametric curve,
$$\frac{dy}{dx} = \frac{Q'(t)}{P'(t)},$$
is positive if the derivatives of both polynomials have the same sign and is negative if they have opposite signs at a point.

Of course, these same observations apply to any parametric curve which is unbounded, but here we are interested solely in polynomial functions.

Based on the above set of observations, we will consider graphically some successive Taylor approximations to various curves that are normally encountered in introductory calculus. First, consider the unit circle

$$x = \cos t,$$
$$y = \sin t.$$

In Figures 3a-d, we show the succession of approximations $\{P_2(t), Q_3(t)\}$, $\{P_4(t), Q_5(t)\}, \{P_6(t), Q_5(t)\}$, and $\{P_6(t), Q_7(t)\}$ on the interval $[-4, 4]$

superimposed over the circle. The approximation shown in Figure 2 was $\{P_8(t), Q_9(t)\}$. While each "polynomial" curve is a better approximation to the circle on the interval $[-\pi, \pi]$, it is surprising how dramatically different the overall behavior of each "polynomial" is away from the circular portion. The interested reader is encouraged to continue this exploration with either a graphing calculator or with any computer graphics package. For instance, you might want to explore what happens if you keep either m or n fixed and increase the other. To what extent does a good approximation to $x(t)$ and a poor approximation to $y(t)$, say, affect the accuracy of the approximation to the actual curve?

A far more challenging example is that of the cycloid

$$x = t - \sin t,$$
$$y = 1 - \cos t.$$

In Figures 4a-d, respectively, we show the successive approximations $\{P_7(t), Q_6(t)\}$, $\{P_9(t), Q_{10}(t)\}$, $\{P_{11}(t), Q_{10}(t)\}$, and $\{P_{15}(t), Q_{14}(t)\}$. Notice how far it is necessary to go to obtain even a poor approximation to the cusp at $t = 2\pi$. The cusp corresponds to a point where both derivatives, $x'(t)$ and $y'(t)$, are zero. Realize, though, that in order to have a polynomial approximation to a cusp, that point corresponds to a multiple root of $Q_n(t)$. Since such multiple roots are extremely sensitive to perturbations, it is not at all surprising how hard it is to obtain a reasonable approximation. In fact, it may be more surprising that we can obtain such an approximation at all.

As a third example, consider the four-cusp hypocycloid

$$x = 3\cos t + \cos 3t,$$
$$y = 3\sin t - \sin 3t,$$

which has cusps at the points $(4, 0), (-4, 0), (0, 4)$, and $(0, -4)$. Each of the cusps corresponds to a multiple root of one of the approximating polynomials, so it is not surprising that it is necessary to use a considerable number of terms to obtain a reasonably good approximation to the curve. We show a sequence of successive approximations, corresponding to the polynomial pairs $\{P_6(t), Q_7(t)\}$, $\{P_8(t), Q_9(t)\}$, $\{P_{14}(t), Q_{15}(t)\}$, $\{P_{16}(t), Q_{17}(t)\}$, $\{P_{18}(t), Q_{17}(t)\}$, $\{P_{18}(t), Q_{19}(t)\}$, $\{P_{20}(t), Q_{19}(t)\}$ and $\{P_{20}(t), Q_{21}(t)\}$, in Figures 5a-h, respectively, on the interval $[-\pi, \pi]$. As before, we encourage the interested reader to experiment with the

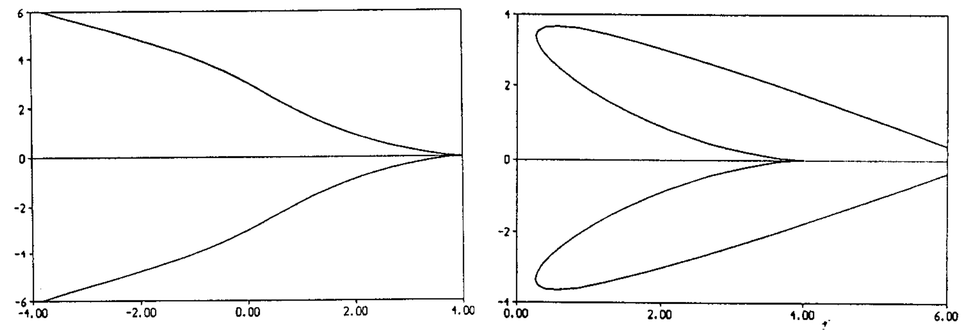

Figure 5: The approximations (a) $\{P_6(t), Q_7(t)\}$, (b) $\{P_8(t), Q_9(t)\}$

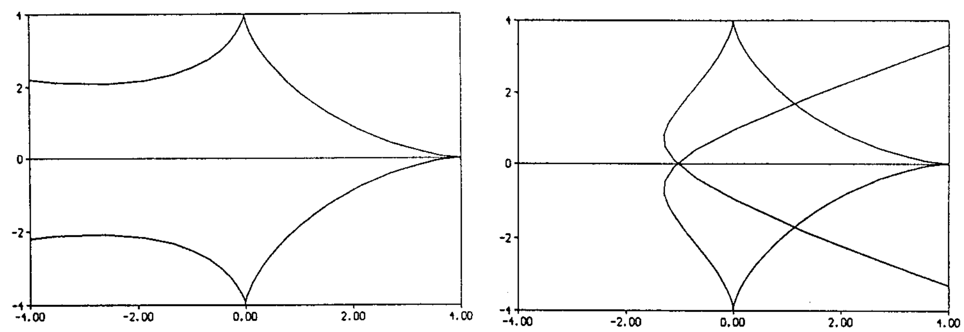

Figure 5: (c) $\{P_{14}(t), Q_{15}(t)\}$, (d) $\{P_{16}(t), Q_{17}(t)\}$

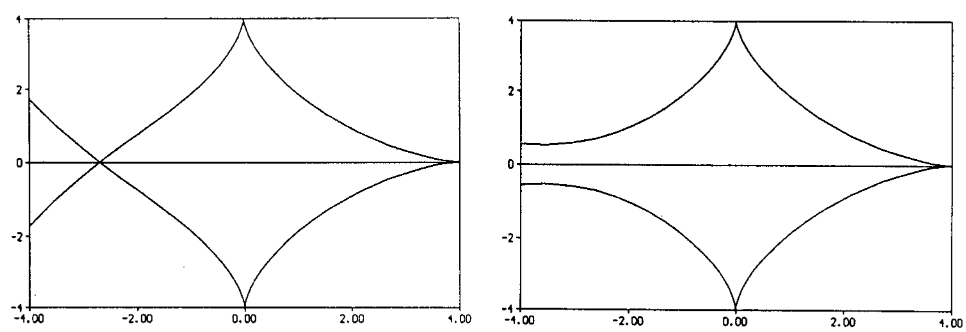

Figure 5: (e) $\{P_{18}(t), Q_{17}(t)\}$, (f) $\{P_{18}(t), Q_{19}(t)\}$

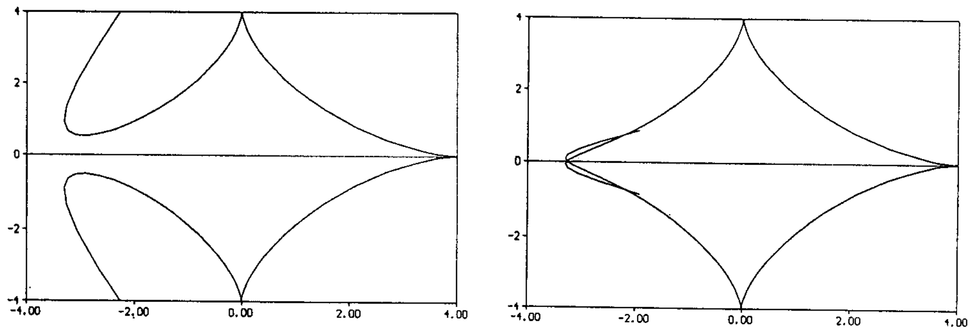

Figure 5: (g) $\{P_{20}(t), Q_{19}(t)\}$ and (h) $\{P_{20}(t), Q_{21}(t)\}$ to be the hypocyloid on the interval $[-\pi, \pi]$.

effects of using a considerably better approximation for one variable than for the other. We also suggest examining the effects of extending the domain for t to considerably larger intervals of values than $-\pi$ to π which was used in producing Figure 5. While the graphs in this article were produced using QuattroPro, comparable graphs can be created using the parametric function display of virtually any graphing calculator or any graphics package intended for calculus.

Finally, we suggest that the above explorations, as well as similar ones based on other well-known parametric curves, are excellent exercises to have students perform in a Calculus III course. Such explorations simultaneously reinforce the notions of Taylor polynomial approximations and parametric curves. Further, the fact that parametric curves that arise are often closed curves means that there is a very specific target on which to focus. In comparison, when we approximate the standard functions, such as e^x or $\sin x$, in rectangular coordinates, we only approximate an indeterminate portion of the graph since the graph is open and so it is much harder to judge when an approximation is adequate. Thus, by conducting the type of study suggested here, students can obtain a better feel for the concept of convergence of a sequence of functions than they might achieve in investigating how successive approximations to e^x or $\sin x$ converge.

We also direct the interested reader to a related exploration [2] of Fourier approximations to polar curves.

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1. Sheldon P. Gordon, "Taylor Polynomial Approximations in Polar Coordinates", *College Mathematics Journal*, Vol. 24, pp. 325-330 (1993).
2. Kenneth S. Gordon, "Fourier Series Approximations in Polar Coordinates", *International Journal of Mathematics Education in Science Technology*, Vol. 26, No. 4 (1995).